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# A Brief Reflection on the Property of Trivialization and Excluded Middle in Classical Logic

**P. M. Tasinaffo**

*Abstract*— **From the two propositional premises There is the Amazon forest and There is no Amazon forest, there is nothing wrong in inferring the conclusions The little duck swims in the blue lagoon or There is life after death. This is the well-known trivialization property of classical logic. It simply states that from a contradiction, anything can be inferred. A contradiction is an argument with two premises and a conclusion in such a way that in the two premises one is exactly the negation of the other. From the trivialization other strange things can be inferred. For example, if It is raining today is a true premise, then, from that, it can be inferred that if There is heat in the desert, then It is raining today. This is the property of material implication, that is, a consequence of trivialization. This article intends to formally explore the mathematical demonstration of the properties of trivialization, material implication and the excluded middle, and briefly discuss these results. These are striking features of classical logic and somewhat is contrary the common sense. Other logic has arisen to circumvent this problem, such as, for example, intuitionist logic and paraconsistent logic.**

*Index Terms*— **Classical logic, expert systems, propositional and predicate calculus, rule of trivialization**

## I. INTRODUCTION

The formal logic, traditionally, concerns the analysis of sentences or propositions and proof with attention to form, in abstraction of its content ([1], [4]). Thoughts, reasoning, inferences, and knowledge, which are psychological processes that pass in the mind of the individual, cannot be the object of investigation. What we can investigate are the fruits of such processes, that is, linguistic expressions ([2]).

As mathematical theory and linguistic theory, mathematical logic has the following characteristics: 1) it is a language with an abstract and formal entity; 2) presents a significant fragment of the natural language (particularized to the expressions of propositions and mathematical theories); 3) an artificial language is conceived, with perfectly defined syntax and semantics; and d) has the task of systematizing valid forms of argumentation.. As a formal language, it introduces a system of rules (inference) to effect valid deductions and reasoning (for example, syllogisms). As a deductive system it presents demonstrative methods that can be of the type: direct, by reduction to the absurd or by cases. The propositional language is composed of: a primitive alphabet (or vocabulary), a grammar or syntax and a semantics. Propositional or propositional logic is the most elementary logic that exists. Thus, the alphabet of the propositional calculus is composed of: 1) propositional letters p, q, r, ... (atoms); 2) primitive propositional connectives, that is, the conjunction ($\wedge$),the disjunction ($\vee$),the negation ($\neg$),the conditional ($\rightarrow$) and the biconditional ($\leftrightarrow$); and 3) the parentheses (,).

The grammar or syntax of propositional language leads to the formation of formulas. The four basic syntax rules of propositional calculus are (see [3] and [7]):

F1. Every propositional letter is a formula;

F2. If $\varphi$ is a formula, then $\neg\varphi$ is a formula;

F3. If $\varphi$, $\psi$ are formulas, then $(\varphi\wedge\psi)$, $(\varphi\vee\psi)$ and $(\varphi\rightarrow\psi)$ are formulas;

F4. Nothing else is formula except those that can be generated by the finite application of rules F2 and F3.

For any formulas $\varphi$ and $\psi$, it is defined the bi-conditional $\leftrightarrow$ as:

$$(\varphi\leftrightarrow\psi) \underset{equivalentto}{\leftrightarrows} ((\varphi\rightarrow\psi)\wedge(\psi\rightarrow\varphi)) \tag{1}$$

Propositional logic is described by a consistent deductive system for propositional language. It consists of: a) a finite list of inference rules; and b) of the concept of derivability ($\vdash$) and bi-derivability ($\dashv\vdash$).These symbols were introduced by Gottlob Frege in 1879. The derivability symbol allows additional or more complex thesis or conclusions to be obtained from rules of primitive inferences. Thus, in a relation between sets of formulas of the type $\varphi_1,\ldots,\varphi_n\vdash\psi$ ( it reads: *from* $\varphi_1,\ldots,\varphi_n$ *is deduced* $\psi$) there is a finite sequence of formulas $\varphi_1,\ldots,\varphi_n,\varphi_{n+1},\ldots,\psi$ which infers the last proposition $\psi$, such that: $\varphi_1,\ldots,\varphi_n$ are the deduction hypothesis; $\varphi_{n+1},\ldots$ are the intermediate thesis; and $\psi$ is said to be the final thesis. In a deductive system when verifying that $\varphi_1,\ldots,\varphi_n\vdash\psi$ is really true, then it can be stated at once that the argument $\varphi_1,\ldots,\varphi_n/\psi$ or $\dfrac{\varphi_1,\cdots\varphi_n}{\psi}$ will also be valid, the latter being non-trivial demonstration result and only obtained from the metatheorem of logic.

Using mathematics, an argument (see [1], [3] and [4]) can be seen as a finite sequence of propositions:

$$\varphi_1,\ldots,\varphi_n,\psi \qquad (n\geq 1) \tag{1.a}$$

In the argument (1.a) read "$\varphi_1,\ldots,\varphi_n$, *therefore* $\psi$".There are other ways to symbolically represent an argument, for example:

$$\frac{\varphi_1,\cdots,\varphi_n}{\psi} \text{ ou } \varphi_1,\cdots,\varphi_n/\psi \tag{1.b}$$

The argument represented in (1.a) or its equivalent forms in (1.b) is basically formed by propositions. The term proposition denotes any declarative sentence which can be said, in a sense, to be true or false. For instance, *Every man is mortal* is a proposition. In this way, the following set of propositions forms an argument:

**P. M. Tasinaffo,** Work supported by the Technological Institute of Aeronautics (ITA). This institution is located in Brazil

$$\frac{\begin{array}{c}\textit{Every man is mortal}\\ \textit{Socrates is a man}\end{array}}{\textit{Therefore, Socrates is mortal}} \tag{2}$$

The above argument is formed by three propositions, be they true or false, the first two being the premises of the argument and the last is the conclusion of the argument. An important concept is the correct or valid argument and incorrect argument definitions. In order to address the above terms in a formal way, we give the following definitions: *valid argument* is one whose conclusion will be true when all premises are simultaneously true e *incorrect argument* is one whose conclusion will be false when all premises are simultaneously true. Let $\Sigma$ be a set of propositions and $\psi$ a proposition. It is said, that $\psi$ is a consequence (logical or semantic) of $\Sigma$, and it is written:

$$\Sigma \models \psi \tag{3}$$

The conclusion $\psi$ is true whenever the propositions of $\Sigma$ are simultaneously true. Thus, when comparing the definitions of valid argument and logical consequence, one can conclude immediately that an argument $\frac{\varphi_1,\cdots,\varphi_n}{\psi}$ is valid if and only if $\varphi_1,\cdots,\varphi_n \models \psi$. Before proceeding to the remainder of this article, we give some definitions of important logical technical terms to assist in a deeper understanding of the mathematical concepts involved in classical logic. These concepts were taken from [8].

*Logic*. A central concern of logic is to discriminate between valid and invalid arguments; and it is intended that formal logical systems, such as the well-known propositional and predicate calculations, provide precise foundations.

*Metalogic*. The Metalogic is the study of the formal properties of formal logical systems. It includes, for example, evidence (or refutations) of its consistency, completeness, or decidability. Ideally a formal logical theory should be consistent, complete, and decidable. Consistency states that only truths must be demonstrable. Completeness states that every true sentence must be a theorem, that is, every truth must be either an axiom or a theorem that follows from the axioms through a formal demonstration process. Decibility is a very general concept that can be applied to sets, terms, formulas, sentences, and also to theories. A set S is decidable if and only if there exists a decision method or process for the set: an algorithm that allows to decide, in a finite number of steps, for each entity s, if s is or not element of S.

*Formal Logic Systems.* Formal logical systems aim to formalize informal arguments, to represent them in precise, rigorous and generalizable terms.

*Axiomatic Logic System.* An axiomatic system of logic includes, alongside one or more rules of inferences, a privileged set of wffs (well-formed formulas), the axioms, that can be used at any point in an argument, and whose truth is unquestionable in the system .

*Logicism.* Logicism is the thesis (suggested by Leibniz, but developed by Frege) that arithmetic is reducible to logic, that arithmetic statements can be expressed in purely logical terms, and that arithmetical theorems can be derived from purely logical axioms.

*Syntax*. Syntax is the study of formal relations between expressions, thus vocabulary, training rules and axioms/ inference rules of a system are called the system syntax.

*Semantics*. Semantics is the study of the relations between linguistic expressions and the non-linguistic objects to which they apply: thus, the interpretation of a system is called the semantics of the system.

*Validity*. A formal argument is syntactically valid in L if and only if its conclusion follows from its premises and from the axioms of L, If there really are axioms, through the inference rules of L. An argument is semantically valid in L if and only if its conclusion is true in all interpretations of L in which all its premises are true. An informal argument is valid if and only if its premises cannot be true and the conclusion false.

*Conceptualization*. Propositional calculus or logic is the most elementary domain of logic and provides the basis for the remainder, which include it. Here we will limit to the propositional calculus of classical logic, which means that: 1) we will only consider the logical operators, the propositional connectives as associated with truth functions.; and that (2) only true (1) and false (0) are taken as truth values.

*Classic Logic*. We can say that classical logic, in line with the Aristotelian tradition, consists of the first order logic, dealing with the logical connectives of negation ($\neg$), conjunction ($\wedge$), disjunction ($\vee$), conditional ($\rightarrow$), biconditional ($\leftrightarrow$), on quantifiers existential ($\exists$) and universal ($\forall$), and on the equality predicate ($=$). Classic propositional calculus, in a sense a subsystem of first-order logic, is also a classical system.

*Basic Principles*. Classical logic is characterized by some basic principles of syntactic and semantic nature. Three of them are fundamental and known as the basic laws of Aristotelian thought: (i) Principle of non-contradiction: a sentence cannot be true or false at the same time; (ii) Principle of the excluded middle: a sentence must be either true or false; and (iii) Principle of identity: every object is identical with itself.

## II. MATHEMATICAL DEMONSTRATIONS

Given these preliminary basic definitions, we present below the formal demonstration of the trivialization rule. It is a relatively simple demonstration that is obtained directly by reduction to absurdity. This demonstration is called (T1). The first two lines of this demonstration are the hypothesis of the deductive system, that is, the contradiction of the system. The third line is a provisional hypothesis. The fourth line of the demonstration is the application of the inference rule of introduction of the conjunction on lines one and two. The fourth line is a contradiction. This means that the provisional hypothesis of line three is absurd, and by exclusion, line five states that it is a truth, the negation of line three. Line six applies the elimination rule of double negation on line five. The law of trivialization is much criticized in modern logics. For example, in the paraconsistent logic, from a contradiction, not everything can be inferred.

The law of trivialization can also be verified by the construction of the truth table. Table 1 presents this demonstration. To verify this, it is enough to apply the valid argument definition, that is, all the lines of the truth table that have simultaneously true premises, it must also have a true conclusion. For the particular case of the law of trivialization, the truth table is also somewhat out of the ordinary. This truth table has the following characteristics: 1) there is no line with simultaneously true premises and true conclusion; 2) there is also no line with simultaneously true premises and false conclusion; 3) the rule of trivialization can be demonstrated by reduction to the absurd (which is also a questionable method within mathematics); and 4) for lack of choice, we are forced to believe that from a contradiction, anything can be inferred.

Table 1. Truth-table of the trivialization rule.

| | | Premise 1 | Premise 2 | Conclusion |
|---|---|---|---|---|
| $\varphi$ | $\psi$ | $\varphi$ | $\neg\varphi$ | $\psi$ |
| 0 | 0 | 0 | 1 | 0 |
| 0 | 1 | 0 | 1 | 1 |
| 1 | 0 | 1 | 0 | 0 |
| 1 | 1 | 1 | 0 | 0 |

T1. $\varphi, \neg\varphi \vdash \psi$ $(\bot)$

Dedution:

| | | | |
|---|---|---|---|
| {1} | 1 | $\varphi$ | H |
| {2} | 2 | $\neg\varphi$ | H |
| {3} | 3 | $\neg\psi$ | {H} |
| {1, 2} | 4 | $\varphi\wedge\neg\varphi$ | 1,2($\wedge^+$) |
| {1, 2} | 5 | $\neg\neg\psi$ | 3-4(RA) |
| {1, 2} | 6 | $\psi$ | 5($\neg\neg^-$) |

*Introduction of Thesis*. Whenever a derivability relation of the form $\varphi_1,\ldots,\varphi_n \vdash \psi$ is established, we can formulate a corresponding derived rule $\frac{\varphi_1,\ldots,\varphi_n}{\psi}$. The disadvantage of this methodology is the considerable increase in the number of rules.

*Principle of Introducing Thesis*. In any line of a deduction, one can insert a thesis, previously already deduced, since the hypothesis necessary for the validation of this thesis are already inserted in the previous lines of the current deduction, and not necessarily in the form of hypothesis. If the previous thesis does not need previous hypothesis, that is, if you are interested in introducing a logical law in the current deduction, so, the thesis does not need hypothesis to be used. Thus, the thesis can be inserted in any line or position of the current deduction and at any time.

The proofs (T2) and (T3) are called respectively the first and second laws of material implication. They are obtained directly from the trivialization rule, using, for this, the principle of the introduction of thesis. The statements (T2) and (T3) also use the conditional introduction rule ($\rightarrow^+$), to eliminate the provisional hypothesis (e.g., see [3]).

T2. $\neg\varphi \vdash \varphi\rightarrow\psi$

Dedution:

| | | | |
|---|---|---|---|
| {1} | 1 | $\neg\varphi$ | H |
| {2} | 2 | $\varphi$ | {H} |
| {1, 2} | 3 | $\psi$ | 1, 2 T1 |
| {1} | 4 | $\varphi\rightarrow\psi$ | 2-3($\rightarrow^+$) |

T3. $\varphi \vdash \psi\rightarrow\varphi$

Dedution:

| | | | |
|---|---|---|---|
| {1} | 1 | $\varphi$ | H |
| {2} | 2 | $\psi$ | {H} |
| {1} | 3 | $\psi\rightarrow\varphi$ | 1-2($\rightarrow^+$) |

An analysis of the deductive systems (T1) and (T3) can really contradict common sense. For example, in (T1) if the premises are true simultaneously *I am here* and *I am not here*, then it is concluded that *The world will end* or *The little duck swims in the blue lagoon*. Notice that in (T1) the propositions $\varphi$ and $\psi$ are completely independent of each other, and, therefore, they can be anything. The deductive system (T3) is also quite curious. He states that if the premise *It is raining today* is true, then the cause of this rain can be anything, for example, *It is heat in the desert* or *God has caused a deluge*. Most interesting of all, it is that laws of material implication are used to deduce the laws of conversion. For example, the law of conversion from the conditional to the disjunction, see demonstration T4, is obtained from the laws of material implication. It is also interesting that (T4) is widely used in digital electronics to convert a conditional into a logic gate type *or*. This demonstration uses the rule of disjunction elimination inference ($\vee^-$) (see [3] for more details).

T4. $\varphi\rightarrow\psi \vdash \neg\varphi\vee\psi$

Dedution:

| | | | |
|---|---|---|---|
| {1} | 1 | $\neg\varphi\vee\psi$ | H |
| {2} | 2 | $\neg\varphi$ | {$H_1$} |
| {2} | 3 | $\varphi\rightarrow\psi$ | 2 T2. |
| {3} | 4 | $\psi$ | {$H_2$} |
| {3} | 5 | $\varphi\rightarrow\psi$ | 4 T3. |
| {1} | 6 | $\varphi\rightarrow\psi$ | 1, 2-3, 4-5($\vee^-$) |

Another questionable property in classical logic is the law of the excluded middle. His proof is given in (T5). This property is also a bit strange, as its truth does not depend on any hypothesis. Therefore, it can be used in any demonstration line without further justification (it is a logical law). Therefore, its use within classical logic is indiscriminate. Intuitionist mathematicians strongly criticize this law. In intuitionist logic, the law of the excluded middle cannot be used indiscriminately.

T5. $\vdash \varphi\vee\neg\varphi$

Dedution:

| | | | |
|---|---|---|---|
| {1} | 1 | $\neg(\varphi\vee\neg\varphi)$ | {H} |
| {2} | 2 | $\varphi$ | {H} |
| {2} | 3 | $\varphi\vee\neg\varphi$ | 2($\vee^+$) |
| {1,2} | 4 | $(\varphi\vee\neg\varphi)\wedge\neg(\varphi\vee\neg\varphi)$ | 1,3($\wedge^+$) |
| {1} | 5 | $\neg\varphi$ | 2-4($RA^*$) |
| {1} | 6 | $\varphi\vee\neg\varphi$ | 5($\vee_2^+$) |
| {1} | 7 | $(\varphi\vee\neg\varphi)\wedge\neg(\varphi\vee\neg\varphi)$ | 1,6($\wedge^+$) |
| | 8 | $\neg\neg(\varphi\vee\neg\varphi)$ | 1-7($RA^*$) |
| | 9 | $\varphi\vee\neg\varphi$ | 8($\neg\neg^-$) |

Notice that the law of the excluded middle can be used indiscriminately to generate the law of conversion given in (T6). This demonstration also uses the disjunction elimination inference rule ($\vee^-$).

Despite these strange results, classical logic is one of the most extraordinary achievements of human science. Classical logic involves propositional logic, first-order logic, and Peano's arithmetic. Its foundations were obtained almost completely between the years of 1847 and 1931. The starting point of classical logic are the works of George Boole of 1847 and the outcome of it is Gödel's famous incompleteness theorem of 1931. The reference [5] provides a rather didactic description for understanding Gödel's incompleteness theorem. In [5] and [6] there is also a description of the automatic methods of proof. Automatic proofing methods, which include forward chaining, backward chaining (in Horn clauses), the full resolution algorithm, and the Tableux method are interesting achievements of Artificial Intelligence made in the sixties of last century. These automatic methods of proof gave rise to what is now called expert systems.

T6. $\varphi \rightarrow \psi \dashv\vdash \neg\varphi \vee \psi$

Dedution:

| | | | |
|---|---|---|---|
| [1] | 1 | $\varphi \rightarrow \psi$ | $H$ |
| | 2 | $\varphi \vee \neg\varphi$ | $(T^{+}, \models \varphi \vee \neg\varphi)$ |
| [2] | 3 | $\varphi$ | $[H_1]$ |
| [1, 2] | 4 | $\psi$ | 1,3(MP) |
| [1, 2] | 5 | $\neg\varphi \vee \psi$ | $4(\vee^{+}_{2})$ |
| [3] | 6 | $\neg\varphi$ | $[H_2]$ |
| [3] | 7 | $\neg\varphi \vee \psi$ | $6(\vee^{+}_{1})$ |
| [1] | 8 | $\neg\varphi \vee \psi$ | 2,3-5,6-7 $(\vee^{-})$ |

## III. CONCLUSION

It is asserted here, without demonstration, that the propositional calculus or the propositional natural deduction system is consistent, complete, and decidable. It is complete considering as truths exactly the valid formulas or tautologies (by virtue of the metatheorem of semantic completeness). It is also decidable, since the construction of a truth table is an algorithmic procedure that, applied to any propositional formula, allows deciding at the end of a finite number of steps whether or not it is valid. The propositional and predicate logic of the first order has a lot of application in artificial intelligence in the design of expert systems.

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**Paulo Marcelo Tasinaffo** is graduated in Mechanical Engineering from the Federal University of Itajubá/MG (UNIFEI, 1996), obtaining a Master's degree in Mechanical Engineering from the same Institution in 1998. In 2003, he received a Doctorate in Space Engineering and Technology from the National Institute of Space Research (INPE) in Brazil. He is currently a full professor of the Aeronautical Technological Institute (ITA) in São José dos Campos/SP also in Brazil, with experience in Aerospace and Computer Engineering and with emphasis on Artificial Neural Networks. The Prof. Tasinaffo works mainly in the following subjects: modeling of non-linear dynamic systems, computational mathematics, artificial intelligence, expert systems, intelligent agents, neural control structures, evolutionary computation and stochastic processes