On Lax-Wendroff-type time integration in high order finite difference schemes

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***Abstract*— In this short note we present a new approach of classical Lax-Wendroff-type time integration for constructing high order finite difference schemes for solving the one-dimensional nonlinear hyperbolic conservation laws. This approach is based on one property of initial equations. The main advantage of this new approach is simplicity, because we use the degrees of Jacoban instead of its higher derivatives. For the first time finite difference schemes with *M*={ 3, 5, 7, 9, 11} temporal and spatial order of accuracy were constructed. Results of selected classical test cases confirm accuracy and non-linear stability of these schemes in the cases of the nonlinear hyperbolic conservation laws.**

***Index Terms*—Lax-Wendroff-type time integration, finite difference scheme, high order of accuracy.**

I. INTRODUCTION

In this paper we describe new approach to using Lax-Wendroff-type time integration [1] in constructing the high order finite difference schemes (FDS). We describe this approach in application to solving the one-dimensional nonlinear hyperbolic conservation law

$$\frac{\partial f}{\partial t}+\frac{\partial g(f)}{\partial x}=0. \qquad (1)$$

Here f is the function, which has to be found from (1), $g(f)$ is the flux function.

At present time, there are two main approaches in time integration [2]. The first approach is using ordinary differential equation (ODE) solver (usually this is Runge-Kutta solver) to solve system of ODE obtained after spatial discretization of (1). Second one is Lax-Wendroff approach, which is based on Taylor expansion on time.

The first approach (Runge-Kutta-type time integration scheme) is used widely in theory of numerical methods for solving of partial derivative equations (PDE) and systems of PDEs [3]. The main advantages of this approach are its clear concept, simplicity in coding, and good stability property (especially if total variation diminishing (TVD) modifications of Runge-Kutta methods [r4] are used). The main lack of this approach is limit of the order of accuracy in TVD Runge-Kutta methods - they cannot exceed fourth order of temporal accuracy [5].

The second approach (Lax-Wendroff-type time integration schemes) is based on classical Lax-Wendroff FDS [1]. The main idea of it is in joint approximation of temporal and spatial derivatives. For this purpose, temporal derivatives in modified equation of FDS are replaced by spatial derivatives using initial PDE and its differential extensions. It leads to significant decreasing of the number of points in the calculation stencil. Another good property is absence of limitations on the order of accuracy - virtually one can construct FDS of any desired order of accuracy. The main drawback of Lax-Wendroff-type time integration schemes is in their complexity, especially for systems of PDEs. In this case one has to operate with multidimensional tensors like $\partial^n g(f)/\partial f^n$ [2]. Therefore at present time there are no Lax-Wendroff-type time integration schemes for equation (1) with higher than fourth order of temporal accuracy.

The aim of this work is to present a way of constructing finite difference schemes for PDE (1) with Lax-Wendroff-type time integration with (any) high order of accuracy. The approach suggested below is deprived of complexity of existing Lax-Wendroff-type time integration schemes because it does not operate with the derivatives of the Jacobian of the PDE (1). At first, one property of the equation (1) is proved. After that, new Lax-Wendroff-type time integration scheme is described. Set of high order (up to eleventh order of temporal and spatial accuracy) FDS for problem (1) is constructed. FInally numerical results show properties of constructed FDSs.

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II. PRELIMINARY

Together with equation (1) we consider its non-conservative form

$$\frac{\partial f}{\partial t}+A\frac{\partial f}{\partial x}=0. \qquad (2)$$

Here $A(f)=\frac{\partial g}{\partial f}$ is the Jacobian of equation (1). We prove one property of equation (1).

Theorem. If f is solution of one-dimensional nonlinear hyperbolic conservation law (1) and all necessary derivatives of f and $g(f)$ exist, then following relations between temporal and spatial derivatives of f

$$\frac{\partial^{k+1} f}{\partial t^{k+1}}=-\frac{\partial^k}{\partial x^k}\left((-A)^k\frac{\partial g}{\partial x}\right), \qquad k=1,\infty. \qquad (3)$$

are correct.

◊We conduct the proof to two steps.

1. We prove first equality of set (3):

$$\frac{\partial^2 f}{\partial t^2}=\frac{\partial}{\partial x}\left(A\frac{\partial g}{\partial x}\right)$$

Its correctness follows from the sequence of equalities, presented below:

$$\left.\begin{array}{l}\dfrac{\partial f}{\partial t}=-\dfrac{\partial g}{\partial x}\\ dg=Adf\end{array}\right\}\Rightarrow\frac{\partial^2 f}{\partial t^2}=\frac{\partial}{\partial t}\left(-\frac{\partial g}{\partial x}\right)=-\frac{\partial}{\partial x}\left(\frac{\partial g}{\partial t}\right)=$$

$$=-\frac{\partial}{\partial x}\left(A\frac{\partial f}{\partial t}\right)=\frac{\partial}{\partial x}\left(A\frac{\partial g}{\partial x}\right)$$

2. We prove, that if equality (3) is correct for $k=n$

$$\frac{\partial^n f}{\partial t^n}=-\frac{\partial^{n-1}}{\partial x^{n-1}}\left((-A)^{(n-1)}\frac{\partial g}{\partial x}\right),$$

then it will be correct for $k=n+1$

$$\frac{\partial^{n+1} f}{\partial t^{n+1}}=-\frac{\partial^{n}}{\partial x^{n}}\left((-A)^{n}\frac{\partial g}{\partial x}\right),$$

First, we examine expression for $\partial^{n+1}f/\partial t^{n+1}$:

$$\frac{\partial^{n+1} f}{\partial t^{n+1}}=\frac{\partial}{\partial t}\left(\frac{\partial^n f}{\partial t^n}\right)=\frac{\partial}{\partial t}\left(-\frac{\partial^{n-1}}{\partial x^{n-1}}\left((-A)^{n-1}\frac{\partial g}{\partial x}\right)\right)=$$

$$=-\frac{\partial^{n-1}}{\partial x^{n-1}}\left(\frac{\partial}{\partial t}\left((-A)^{n-1}\frac{\partial g}{\partial x}\right)\right)= \qquad (4)$$

$$=-\frac{\partial^{n-1}}{\partial x^{n-1}}\left(\frac{\partial}{\partial t}\left((-A)^{n-1}\right)\frac{\partial g}{\partial x}+(-A)^{n-1}\frac{\partial^2 g}{\partial t\partial x}\right).$$

Further, we consider the summands in the brackets in right part of (4):

$$\frac{\partial}{\partial t}\left((-A)^{n-1}\right)=\frac{\partial}{\partial f}\left((-A)^{n-1}\right)\frac{\partial f}{\partial t}=$$

$$=-\frac{\partial}{\partial f}\left((-A)^{n-1}\right)\frac{\partial g}{\partial x}=-\frac{\partial}{\partial f}\left((-A)^{n-1}\right)A\frac{\partial f}{\partial x}= \qquad (5)$$

$$=-\frac{\partial}{\partial x}\left((-A)^{n-1}\right)A,$$

$$\frac{\partial^2 g}{\partial t\partial x}=\frac{\partial}{\partial x}\left(\frac{\partial g}{\partial t}\right)=\frac{\partial}{\partial x}\left(A\frac{\partial f}{\partial t}\right)=-\frac{\partial}{\partial x}\left(A\frac{\partial g}{\partial x}\right). \qquad (6)$$

Non-conservative form (2) of (1) was used intensively at chains (5, 6).

To complete the proof of step 2, we substitute relations (5, 6) into (4):

$$\frac{\partial^{n+1} f}{\partial t^{n+1}}=$$

$$=-\frac{\partial^{n-1}}{\partial x^{n-1}}\left(\frac{\partial}{\partial x}\left((-A)^{n-1}\right)\left(-A\frac{\partial g}{\partial x}\right)+(-A)^{n-1}\frac{\partial}{\partial x}\left(-A\frac{\partial g}{\partial x}\right)\right)=$$

$$=-\frac{\partial^{n-1}}{\partial x^{n-1}}\left(\frac{\partial}{\partial x}\left((-A)^{n-1}(-A)\frac{\partial g}{\partial x}\right)\right)=-\frac{\partial^{n-1}}{\partial x^{n-1}}\left(\frac{\partial}{\partial x}\left((-A)^{n}\frac{\partial g}{\partial x}\right)\right)$$

Finally, the correctness of theorem follows immediately from step 1 and step 2 on an induction. ◊

III. Construction of the Schemes

In domain $\{0\le x\le 1, 0\le t\le T,\}$ we consider uniform grid

$$\omega_{\tau h}=\left\{x_i=ih\,|\,i=1,N_x;t_j=n\tau\,|\,n=1,N_t\right\},$$

where τ,h are uniform (for simplicity) temporal and spatial steps respectively. We denote the point value of any grid function $\varphi(t_n,x_i)=\varphi(n\tau,ih)$ as φ_i^n.

Following classical Lax-Wendroff-type time integration scheme [1, 2], we obtain semi-discrete form of initial equation (1). For this purpose we consider following expression for derivative $\partial f/\partial t$:

$$\frac{\partial f}{\partial t}=\frac{f_i^{n+1}-f_i^n}{\tau}-\sum_{k=1}^{\infty}\frac{\tau^k}{(k+1)!}\frac{\partial^{k+1}f}{\partial t^{k+1}}. \qquad (7)$$

Such representation follows from Taylor expansion of the finite difference $\left(f_i^{n+1}-f_i^n\right)/\tau$ around the point (t_n,x_i). Substituting (7) in (1), we obtain semi-discrete form of initial equation (1):

$$\frac{f_i^{n+1}-f_i^n}{\tau}+\frac{\partial g}{\partial x}-\sum_{k=1}^{\infty}\frac{\tau^k}{(k+1)!}\frac{\partial^{k+1}f}{\partial t^{k+1}}=0. \qquad (8)$$

Further, we replace temporal derivatives in (8), using (3):

$$\frac{f_i^{n+1}-f_i^n}{\tau}+\frac{\partial g}{\partial x}+\sum_{k=1}^{\infty}\frac{\tau^k}{(k+1)!}\frac{\partial^k}{\partial x^k}\left((-A)^k\frac{\partial g}{\partial x}\right)=0. \qquad (9)$$

We remark, that equation (9) is completely equivalent to the initial one (1). The difference between them will arise, when finite number of terms in sum in (9) is taken into account. We have to consider $(M-1)$ first terms in sum to achieve M order of temporal accuracy.

We derive from equation (9) original two-step approach for constructing finite difference scheme with desired order of temporal and spatial accuracy M for solving (1):

1. We reconstruct finite difference approximation of derivative $\partial g/\partial x$ with M order of accuracy

$$\varphi_i=L_M^1\left(g_i^n\right)=\left.\frac{\partial g(f)}{\partial x}\right|_i+0\left(h^M\right). \qquad (10)$$

Hereinafter L_M^k is some difference operator, which approximates derivative $\partial^k/\partial x^k$ with M spatial order of accuracy.

2. We calculate f_i^{n+1} on the new temporal layer using relation

$$f_i^{n+1}=f_i^n-\tau\varphi_i-\sum_{k=1}^{M-1}\frac{\tau^{k+1}}{(k+1)!}L_{M-k}^k\left((-A_i)^k\varphi_i\right). \qquad (11)$$

Any type of reconstruction (from simplest centered finite differences [6] up to modern used widely WENO reconstruction and its modifications [2,7,8]) may be used at first step for definition operator L_M^1. Using simplest centered finite differences for definition of set of operators $\left\{L_{M-k}^k\,\middle|\,k=2,M-1\right\}$ at the second step is sufficient for stable calculations [2].

We remark, that no higher derivatives of Jacobian but only its degrees are used in suggested approach. It leads to significant simplification of code in the case of vector equation (1). So our approach is free of the main lack of existing Lax-Wendroff-type time integration schemes.

IV. Numerical Results

We construct FDSs with $M = \{3, 5, 7, 9, 11\}$ temporal and spatial order of accuracy for the numerical calculations. We use centered finite differences at the all steps of FDS. We add the dissipative term of $(M+1)$ spatial order of accuracy at the second step (11) for suppressing high-frequence oscillations in solution:

$$\nu_i = h^{M+1} L_2^1 \left(\alpha_{i+1/2} L_2^M \left(f_i^n \right) \right) \quad (12)$$

We use well-known minmod-limiter\cite [6] for definition the value of $\alpha_{i+1/2}$. It provides non-increasing of total variation of numerical solution.

We checked stability of FDSs, used in this session. Spectral analysis in linear case and analysis of modified equation showed conditional stability of all used FDSs with classical Courant-Friedrichs-Lewy condition

$$Cu = \frac{\tau |\lambda_{\max}|}{h} < 1$$

Here $\lambda_{\max}$ is maximum velocity of disturbance propagation in each solved problem. We conduct all calculation with $Cu = 0.5$ in this session.

We consider the accuracy test for nonlinear scalar Burgers equation [2]:

$$\frac{\partial f}{\partial t} + \frac{\partial}{\partial x}\left(\frac{f^2}{2} \right) = 0.$$

with initial condition $f(0,x) = 0.5 + \sin(\pi x)$ and 2-periodical boundary condition. We conduct calculation up to time $t = 0.5/\pi$, when solution is still smooth. Relations of L_1 norm of approximation error to the number of grid point are presented in Fig. 1.

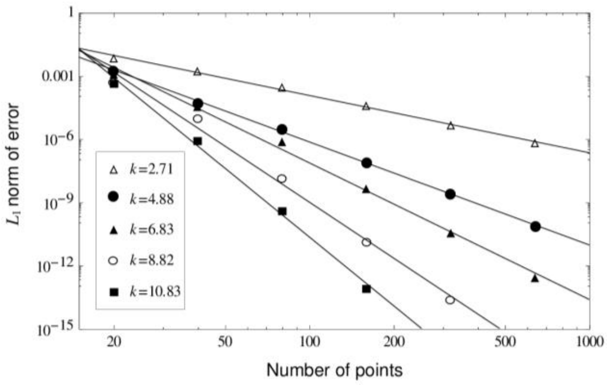

Fig. 1. Approximation error of solution of Burgers equation

Hereinafter k is estimation of order of approximation, defined by the calculation results. We can see that all FDSs aspire to their order of accuracy. We notice, that introduction of a dissipative term (12) reduces insignificantly an accuracy order.

V. Concluding Remark

We have developed a way to construct FDSs for PDE (1) with Lax-Wendroff-type time integration with (any) high order of temporal accuracy. This approach is based on classical Lax-Wendroff-type time integration scheme and one property of hyperbolic PDE (3), proved in this paper. For the first time FDSs with $M = \{3, 5, 7, 9, 11\}$ temporal and spatial order of accuracy were constructed on the base of suggested approach. Numerical tests were conducted successfully to verify the accuracy for the case of the scalar hyperbolic equation.

References

[1] P.D. Lax and B. Wendroff, "Systems of conservation laws", *Communications in Pure and Applied Mathematics* 13, 1960, pp.217-237.
[2] Qiu and C.-W. Shu, "Finite difference WENO schemes with Lax-Wendroff type time discretization", *SIAM Journal on Scientific Computing*, 24, 2003, pp. 2185-2198
[3] C.-W. Shu, "Total-Variation-Diminishing time discretizations", *SIAM Journal on Scientifc and Statistical Computing*, 9, 1988, pp.1073-1084.
[4] B. Cockburn, C. Johnson, C.-W. Shu and E. Tadmor, "Essentially non-oscillatory and weighted essentially non-oscillatory schemes for hyperbolic conservation laws", in: *Advanced Numerical Approximation of Nonlinear Hyperbolic Equations*", (Editor: A. Quarteroni), Lecture Notes in Mathematics, volume 1697, Springer, 1998, pp.325-432.
[5] S.J. Ruuth and R.J. Spiteri, "Two barriers on strong-stability preserving time discretization methods", *Journal of Scientific Computing* 17, 2002, pp. 211-220.
[6] *Encyclopedia of Computational Mechanics* Volume 1 Fundamentals /Editors Erwin Stein, Rene de Borst, Thomas J. R. Hughes, WILEY, 2004
[7] Y. Jiang, C.-W. Shu and M. Zhang "An alternative formulation of finite difference WENO schemes with Lax-Wendroff time discretization for conservation laws", unpublished
[8] J. Qiu "Hermite WENO schemes with lax-wendroff type time discretizations for hamilton-jacobi equations," *Journal of Computational Mathematics*, 25, 2007, pp. 131-144.

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