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A Common Fixed Point Theorem for Four Mappings in $\boldsymbol{k}$ −Metric Spaces

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***Abstract*— The only difference between ordinary metric and k-metric is in the triangle inequality. In this paper we have shown that instead of this difference a common fixed point theorem for four mappings can be obtained. The theorem presented generalizes an existing result from the literature.**

***Index Terms*—Mapping, Fixed point theorem. k- Metric spaces.**

I. INTRODUCTION

In 2012, H. Pajoohesh [1] introduced the concept of k-metric spaces. In this paper we generalize a result of [3] in the language of k-metric spaces. As in [1] a k-metric, where k is a real number ≥ 1, on a nonempty set X is a mapping $d: X \times X \to \mathbb{R}$ such that

(i) $d(x,y) \geq 0 \ \forall \ x,y \in X$,

(ii) $d(x,y) = 0 \Leftrightarrow x = y$,

(iii) $d(x,y) = d(y,x) \ \forall \ x,y \in X$,

(iv) $d(x,y) \leq k(d(x,z) + d(z,y)) \ \forall \ x,y,z \in X$.

The ordered pair (X,d) is called a k-metric space.

Let us consider the mapping $d: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by $d(x,y) = (x-y)^2 \quad \forall x,y \in \mathbb{R}$. The fact $(a+b)^2 \leq 2(a^2+b^2) \quad \forall a,b \in \mathbb{R}$ ensures that the mapping d enjoys all the properties of being a k-metric for $k = 2$.

From the definition and the example, just given above, it is clear that every metric is a k-metric $(k=1)$, but a k-metric may not be a metric and every k-metric is an l-metric, where $l \geq k$.

Open balls, closed balls, diameter of non empty sets, open sets (A subset O of a k-metric space (X,d) is said to be open in (X,d) if $\forall \ x \in O \ \exists \ \varepsilon > 0$ such that the open ball $B_d(x,\varepsilon) \subset O$.), closed sets, closure and interior of a set, convergence of a sequence, Cauchy sequence, completeness of k-metric spaces are defined as in case of metric spaces. It is also seen that every k-metric space is first countable and T_4. [2], [4], [5] motivate to work on this field.

II. A COMMON FIXED POINT THEOREM FOR FOUR MAPPINGS

In this section we prove a common fixed point theorem for four self mappings on a complete k-metric space. For that we need following definitions. As in [3]

Definition 1 *Let f and g be self mappings on a set X. A point $x \in X$ is called a* **coincidence point** *of f and g if $fx = gx = w$, where w is called a point of coincidence of f and g.*

Definition 2 *Two self mappings f and g on a set X are said to be* **weakly compatible** *if f and g commute at their coincidence points that is if $fx = gx$ for some $x \in X$, then $fgx = gfx$.*

Theorem 1 *Let (X,d) be a complete k-metric space. Suppose that f, g ,F and G are self mappings on X satisfying the following conditions :*

(a) $f(X) \subseteq g(X)$ and $F(X) \subseteq G(X)$

(b) $\exists \ \delta > 0, \ L \geq 0$ satisfying $\delta k + kL(1+k) < 1$ such that

$$d(Fx,fy) \leq \delta M(x,y) + Lmin[d(gx,Fx), d(gx,fy)] \quad \forall x,y \in X, \text{ where}$$

$$M(x,y) = max[d(gx,Gy), d(gx,Fx), d(Gy,fy), \frac{1}{2k}\{d(gx,fy) + d(Gy,Fx)\}]$$

(c) $f(X)$ or $g(X)$ is closed.

If $\{f,G\}$ and $\{g,F\}$ are weakly compatible, then f, g, F and G have a unique common fixed point in X.

Proof. Suppose that x_0 is an arbitrary point in X. Since, $f(X) \subseteq g(X)$ and $F(X) \subseteq G(X)$, one may construct a sequence$\{y_n\}$ in X satisfying $y_n = Fx_n = Gx_{n+1}$ and $y_{n+1} = fx_{n+1} = gx_{n+2}$ for all $n \in \mathbb{N} \cup \{0\}$. By the given condition,

$$d(Fx_n, fx_{n+1}) \leq \delta M(x_n, x_{n+1}) + Lmin\{d(gx_n, Fx_n), d(gx_{n+1}, fx_{n+1})\} \quad .$$

Since,

$$\begin{aligned}
M(x_n, x_{n+1}) &= max\{d(gx_n, Gx_{n+1}), \ d(gx_n, Fx_n), \ d(Gx_{n+1}, fx_{n+1}), \\
&\qquad \frac{1}{2k}[d(gx_n, fx_{n+1}) + d(Gx_{n+1}, Fx_n)]\} \\
&= max\{d(y_{n-1}, \ y_n), d(y_{n-1}, y_n), \ d(y_n, y_{n+1}), \\
&\qquad \frac{1}{2k}[d(y_{n-1}, y_{n+1}) + d(y_n, y_n)]\} \\
&\leq max\{d(y_{n-1}, y_n), \ d(y_n, y_{n+1}), \ \frac{k}{2k}[d(y_{n-1}, y_n) + d(y_n, y_{n+1})]\} \\
&= max\{d(y_{n-1}, y_n), \ d(y_n, y_{n+1}), \ \frac{1}{2}[d(y_{n-1}, y_n) + d(y_n, y_{n+1})]\}
\end{aligned}$$

and

$$min\{d(gx_n, Fx_n), d(gx_n, fx_{n+1})\} = min\{d(y_{n-1}, y_n), d(y_{n-1}, y_{n+1})\}$$

we obtain

$$\begin{aligned}
d(y_n, y_{n+1}) &= d(Fx_n, fx_{n+1}) \\
&\leq \delta \quad max\{d(y_{n-1}, y_n), d(y_n, y_{n+1})\} \\
&\quad +L \quad min\{d(y_{n-1}, y_n) + d(y_{n-1}, y_{n+1})\}
\end{aligned}$$

We split-up the proof into the following cases.

Case: 1 If,

$$max\{d(y_{n-1}, y_n), d(y_n, y_{n+1})\} = d(y_{n-1}, y_n)$$
$$min\{d(y_{n-1}, y_n), d(y_{n-1}, y_{n+1})\} = d(y_{n-1}, y_n)$$

Then

$$\begin{aligned}
d(y_n, y_{n+1}) &\leq \delta \quad d(y_{n-1}, y_n) + L \quad d(y_{n-1}, y_n) \\
&= (\delta + L) d(y_{n-1}, y_n)
\end{aligned}$$

Let $k_1 = (\delta + L)$. Since $\delta k + kL(1+k) < 1$, we have $k_1 < \frac{1}{k}$ and $d(y_n, y_{n+1}) < k_1 d(y_{n-1}, y_n)$

Case: 2 If,

$$max\{d(y_{n-1},y_n),\ d(y_n,y_{n+1})\} = d(y_{n-1},y_n)$$
$$min\{d(y_{n-1},y_n), d(y_{n-1},y_{n+1})\} = d(y_{n-1},y_{n+1})$$

Then

$$d(y_n,y_{n+1}) \le \delta\ d(y_{n-1},y_n) + L\ d(y_{n-1},y_{n+1})$$
$$\le \delta d(y_{n-1},y_n) + Lk\ d(y_{n-1},y_n) + L\ k\ d(y_n,y_{n+1})$$
$$\Rightarrow (1-Lk)d(y_n,y_{n+1}) \le (\delta + kL)d(y_{n-1},y_n)$$
$$\Rightarrow d(y_n,y_{n+1}) \le \frac{\delta + kL}{1-kL} d(y_{n-1},y_n)$$

Let $\frac{\delta+kL}{1-kL} = k_2$, since $\delta k + kL(1+k) < 1$,

$k_2 < \frac{1}{k}$ and $d(y_n,y_{n+1}) \le k_2 d(y_{n-1},y_n)$

Case: 3 If

$$max\{d(y_{n-1},y_n), d(y_n,y_{n+1})\} = d(y_n,y_{n+1})$$
$$min\{d(y_{n-1},y_n), d(y_{n-1},y_{n+1})\} = d(y_{n-1},y_n)$$

Then

$$d(y_n,y_{n+1}) \le \delta\ d(y_n,y_{n+1}) + L\ d(y_{n-1},y_n)$$
$$\Rightarrow (1-\delta)d(y_n,y_{n+1}) \le L\ d(y_{n-1},y_n)$$
$$\Rightarrow d(y_n,y_{n+1}) \le \frac{L}{1-\delta}\ d(y_{n-1},y_n)$$

Let $\frac{L}{1-\delta} = k_3$, since $\delta k + kL(1+k) < 1$,

$k_3 < 1$ and $d(y_n,y_{n+1}) \le k_3 d(y_{n-1},y_n)$.

Case: 4 If

$$max\{d(y_{n-1},y_n), d(y_n,y_{n+1})\} = d(y_n,y_{n+1})$$
$$min\{d(y_{n-1},y_n), d(y_{n-1},y_{n+1})\} = d(y_{n-1},y_{n+1})$$

Then

$$d(y_n,y_{n+1}) \le \delta d(y_n,y_{n+1}) + L d(y_{n-1},y_{n+1})$$
$$\le \delta d(y_n,y_{n+1}) + Lk d(y_{n-1},y_n) + Lk d(y_n,y_{n+1})$$
$$\Rightarrow (1-\delta-Lk)d(y_{n+1},y_n) \le Lk d(y_{n-1},y_n)$$
$$\Rightarrow d(y_n,y_{n+1}) \le \frac{kL}{1-\delta-kL} d(y_n,y_{n-1})$$

Let $\frac{kL}{1-\delta-kL} = k_4$, since $\delta k + kL(1+k) < 1$,

$k_4 < \frac{1}{k}$ and $d(y_n,y_{n+1}) \le k_4 d(y_{n-1},y_n)$

Let $h = max\{k_1,k_2,k_3,k_4\}$ then $h < \frac{1}{k}$ and $d(y_{n+1},y_n) \le hd(y_n,y_{n-1}) \le h^n d(y_0,y_1)$.

Now for $n > m$

$$d(y_m,y_n) \le kd(y_m,y_{m+1}) + kd(y_{m+1},y_n)$$
$$\le kd(y_m,y_{m+1}) + k^2 d(y_{m+1},y_{m+2}) + k^2 d(y_{m+2},y_n)$$
$$\le kd(y_m,y_{m+1})$$
$$+ k^2 d(y_{m+1},y_{m+2}) + \ldots + k^{n-m} d(y_{n-1},y_n)$$
$$\le kh^m d(y_0,y_1)$$
$$+ k^2 h^{m+1} d(y_0,y_1) + \ldots + k^{n-m} h^{n-1} d(y_0,y_1)$$
$$= h^m (k + k^2 h + k^3 h^2 + \ldots + k^{n-m} h^{n-m-1}) d(y_0,y_1)$$
$$= h^m k(1 + (kh) + (kh)^2 + \ldots + (kh)^{n-m-1}) d(y_0,y_1)$$
$$< \frac{h^m k}{1-kh} (\text{since} kh < 1)$$
$$\to 0 \text{as} m \to \infty$$

Therefore $\{y_n\}$ is a Cauchy sequence in (X,d), since (X,d) is complete, there exist $z \in X$ such that $\lim_{n\to\infty} y_n = z$.

Assume that $g(X)$ is closed, therefore there exist a point $u \in X$ such that $z = gu$

Now we have,

$$d(z,Fu)$$
$$\le k\ d(z,y_{n+1}) + k\ d(y_{n+1,Fu})$$
$$= k\ [d(z,y_{n+1}) + d(fx_{n+1},Fu)]$$
$$\le k\ [d(z,y_{n+1})$$
$$+ \delta\ max\{d(gu,Gx_{n+1}), d(gu,fu), d(Gx_{n+1},fx_{n+1}),$$
$$\frac{1}{2k}[d(gu,fx_{n+1}) + d(Gx_{n+1},Fu)]\}$$
$$+ L\ min\{d(gu,Fu), d(gu,fx_{n+1})\}]$$
$$= k\ [d(z,y_{n+1})$$
$$+ \delta\ max\{d(z,y_n), d(z,fu), d(y_n,y_{n+1}),$$
$$\frac{1}{2k}[d(z,y_{n+1}) + d(y_n,Fu)]\}$$
$$+ L\ min\{d(z,Fu), d(z,y_{n+1})\}]$$
$$\le k\ d(z,y_{n+1})$$
$$+ \delta\ k\ max\{d(z,y_n), d(z,fu), [k\ d(y_n,z)$$
$$+ k\ d(z,y_{n+1})],$$
$$\frac{1}{2k}[d(z,y_{n+1}) + k\ d(y_n,z) + k\ d(z,Fu)]\}$$
$$+ L\ min\{d(z,Fu), d(z,y_{n+1})\}$$

Taking limit as $n \to \infty$ we get,

$d(z,fu) \le \delta k d(z,Fu)$ [by the given condition $\delta k < 1$]

Therefore, $d(z,Fu) = 0 \Rightarrow z = Fu$

Since F and g are weakly compatible, we obtain that,

$gFu = Fgu$

$\Rightarrow gz = Fz$

Since $F(X) \subseteq G(X)$, there exist $v \in X$ such that $Gv = z$

Applying the given condition we get,

$$d(z,fv) = d(Fu,fv)$$
$$\le \delta\ max\{d(gu,Gv), d(gu,Fu), d(Gv,fv),$$
$$\frac{1}{2k}[d(gu,fv) + d(Gv,Fu)]\}$$
$$+ L\ min\{d(gu,Fu), d(gu,fv)\}$$
$$\therefore d(z,fv) \le \delta d(z,fv)$$
$$\Rightarrow d(z,fv) = 0 \quad (\because \delta k \le 1)$$
$$\therefore z = fv = Gv$$

Since G and f are weakly compatible, we obtain that

$$fGv = Gfv$$
$$\Rightarrow fz = Gz$$

$$d(Fz,z) = d(Fz,fv)$$
$$\le \delta max\{d(gz,Gv), d(gz,Fz), d(Gv,fv), \frac{1}{2k}[d(gz,fv)$$
$$+ d(Gv,Fz)]\}$$
$$+ Lmin\{d(gz,Fz), d(Gv,Fz)\}$$
$$= \delta max\{d(Fz,z), d(Fz,Fz), d(z,z), \frac{1}{2k}[d(Fz,z)$$
$$+ d(z,Fz)]\}$$
$$+ Lmin\{d(Fz,Fz), d(Fz,z)\}$$
$$= \delta d(Fz,z)$$
$$\Rightarrow d(Fz,z) = 0$$

So, $gz = Fz = z$. Similarly we get,

$$d(z,fz) = d(Fz,fz)$$
$$\le \delta max\{d(gz,Gz), d(gz,Fz), d(Gz,fz), \frac{1}{2k}[d(gz,fz)$$
$$+ d(Gz,Fz)]\}$$
$$+ Lmin\{d(gz,Fz), d(gz,fz)\}$$
$$= \delta max\{d(z,fz), d(z,z), d(fz,fz), \frac{1}{2k}[d(z,fz)$$
$$+ d(fz,z)]\}$$
$$+ Lmin\{d(z,z), d(z,fz)\}$$
$$= \delta d(z,fz)$$
$$\Rightarrow d(z,fz) = 0$$

So, $Gz = fz = z$ and therefore z is common fixed point of f,g,F and G.

Uniqueness of such common fixed point:

Let $p \in X$ be also a common fixed point of f, g, F and G. Again applying the given condition we get,

$$d(z,p) = d(Fz, fp)$$
$$\leq \delta max\{d(gz,Gp), d(gz,Fz), d(Gp,fp), \frac{1}{2k}[d(gz,fp) + d(Gp,Fz)]\} + Lmin\{d(gz,Fz), d(gz,fp)\}$$
$$= \delta max\{d(z,p), d(z,z), d(p,p), \frac{1}{2k}[d(z,p) + d(p,z)]\} + Lmin\{d(z,z), d(z,p)\}$$
$$= \delta d(z,p)$$

i.e. $d(z,p) \leq \delta d(z,p)$

This implies that $d(z,p) = 0$ and so $z = p$

Hence f, g, F and G have a unique common fixed point in X.

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