

A Common Fixed Point Theorem for Four Mappings in k –Metric Spaces

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Abstract— The only difference between ordinary metric and k -metric is in the triangle inequality. In this paper we have shown that instead of this difference a common fixed point theorem for four mappings can be obtained. The theorem presented generalizes an existing result from the literature.

Index Terms—Mapping, Fixed point theorem. k - Metric spaces.

I. INTRODUCTION

In 2012, H. Pajoohesh [1] introduced the concept of k -metric spaces. In this paper we generalize a result of [3] in the language of k -metric spaces. As in [1] a k -metric, where k is a real number ≥ 1 , on a nonempty set X is a mapping $d: X \times X \rightarrow \mathbb{R}$ such that

- (i) $d(x, y) \geq 0 \forall x, y \in X$,
- (ii) $d(x, y) = 0 \Leftrightarrow x = y$,
- (iii) $d(x, y) = d(y, x) \forall x, y \in X$,
- (iv) $d(x, y) \leq k(d(x, z) + d(z, y)) \forall x, y, z \in X$.

The ordered pair (X, d) is called a k -metric space.

Let us consider the mapping $d: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ defined by $d(x, y) = (x - y)^2 \forall x, y \in \mathbb{R}$. The fact $(a + b)^2 \leq 2(a^2 + b^2) \forall a, b \in \mathbb{R}$ ensures that the mapping d enjoys all the properties of being a k -metric for $k = 2$.

From the definition and the example, just given above, it is clear that every metric is a k -metric ($k = 1$), but a k -metric may not be a metric and every k -metric is an l -metric, where $l \geq k$.

Open balls, closed balls, diameter of non empty sets, open sets (A subset O of a k -metric space (X, d) is said to be open in (X, d) if $\forall x \in O \exists \varepsilon > 0$ such that the open ball $B_d(x, \varepsilon) \subset O$), closed sets, closure and interior of a set, convergence of a sequence, Cauchy sequence, completeness of k -metric spaces are defined as in case of metric spaces. It is also seen that every k -metric space is first countable and T_4 . [2], [4], [5] motivate to work on this field.

II. A COMMON FIXED POINT THEOREM FOR FOUR MAPPINGS

In this section we prove a common fixed point theorem for four self mappings on a complete k -metric space. For that we need following definitions. As in [3]

Definition 1 Let f and g be self mappings on a set X . A point $x \in X$ is called a **coincidence point** of f and g if $fx = gx = w$, where w is called a point of coincidence of f and g .

Definition 2 Two self mappings f and g on a set X are said to be **weakly compatible** if f and g commute at their coincidence points that is if $fx = gx$ for some $x \in X$, then $fgx = gfx$.

Theorem 1 Let (X, d) be a complete k -metric

space. Suppose that f, g, F and G are self mappings on X satisfying the following conditions :

- (a) $f(X) \subseteq g(X)$ and $F(X) \subseteq G(X)$
- (b) $\exists \delta > 0, L \geq 0$ satisfying $\delta k + kL(1 + k) < 1$ such that

$$d(Fx, fy) \leq \delta M(x, y) + L \min\{d(gx, Fx), d(gx, fy)\} \quad \forall x, y \in X, \text{ where}$$

$$M(x, y) = \max\{d(gx, Gy), d(gx, Fx), d(Gy, fy), \frac{1}{2k}\{d(gx, fy) + d(Gy, Fx)\}\}$$

- (c) $f(X)$ or $g(X)$ is closed.

If $\{f, G\}$ and $\{g, F\}$ are weakly compatible, then f, g, F and G have a unique common fixed point in X .

Proof. Suppose that x_0 is an arbitrary point in X . Since, $f(X) \subseteq g(X)$ and $F(X) \subseteq G(X)$, one may construct a sequence $\{y_n\}$ in X satisfying $y_n = Fx_n = Gx_{n+1}$ and $y_{n+1} = fx_{n+1} = gx_{n+2}$ for all $n \in \mathbb{N} \cup \{0\}$. By the given condition,

$$d(Fx_n, fx_{n+1}) \leq \delta M(x_n, x_{n+1}) + L \min\{d(gx_n, Fx_n), d(gx_{n+1}, fx_{n+1})\}$$

Since,

$$M(x_n, x_{n+1}) = \max\{d(gx_n, Gx_{n+1}), d(gx_n, Fx_n), d(Gx_{n+1}, fx_{n+1}), \frac{1}{2k}[d(gx_n, fx_{n+1}) + d(Gx_{n+1}, Fx_n)]\}$$

$$= \max\{d(y_{n-1}, y_n), d(y_{n-1}, y_n), d(y_n, y_{n+1}), \frac{1}{2k}[d(y_{n-1}, y_{n+1}) + d(y_n, y_n)]\}$$

$$\leq \max\{d(y_{n-1}, y_n), d(y_n, y_{n+1}), \frac{k}{2k}[d(y_{n-1}, y_n) + d(y_n, y_{n+1})]\}$$

$$= \max\{d(y_{n-1}, y_n), d(y_n, y_{n+1}), \frac{1}{2}[d(y_{n-1}, y_n) + d(y_n, y_{n+1})]\}$$

and

$$\min\{d(gx_n, Fx_n), d(gx_n, fx_{n+1})\} = \min\{d(y_{n-1}, y_n), d(y_{n-1}, y_{n+1})\}$$

we obtain

$$d(y_n, y_{n+1}) = d(Fx_n, fx_{n+1}) \leq \delta \max\{d(y_{n-1}, y_n), d(y_n, y_{n+1})\} + L \min\{d(y_{n-1}, y_n) + d(y_{n-1}, y_{n+1})\}$$

We split-up the proof into the following cases.

Case: 1 If,

$$\max\{d(y_{n-1}, y_n), d(y_n, y_{n+1})\} = d(y_{n-1}, y_n)$$

$$\min\{d(y_{n-1}, y_n), d(y_{n-1}, y_{n+1})\} = d(y_{n-1}, y_n)$$

Then

$$d(y_n, y_{n+1}) \leq \delta d(y_{n-1}, y_n) + L d(y_{n-1}, y_n) = (\delta + L)d(y_{n-1}, y_n)$$

Let $k_1 = (\delta + L)$. Since $\delta k + kL(1 + k) < 1$, we have

$$k_1 < \frac{1}{k} \text{ and } d(y_n, y_{n+1}) < k_1 d(y_{n-1}, y_n)$$

Case: 2 If,

$$\begin{aligned} \max\{d(y_{n-1}, y_n), d(y_n, y_{n+1})\} &= d(y_{n-1}, y_n) \\ \min\{d(y_{n-1}, y_n), d(y_n, y_{n+1})\} &= d(y_{n-1}, y_{n+1}) \end{aligned}$$

Then

$$\begin{aligned} d(y_n, y_{n+1}) &\leq \delta d(y_{n-1}, y_n) + L d(y_{n-1}, y_{n+1}) \\ &\leq \delta d(y_{n-1}, y_n) + Lk d(y_{n-1}, y_n) + Lk d(y_n, y_{n+1}) \\ &\Rightarrow (1 - Lk)d(y_n, y_{n+1}) \leq (\delta + kL)d(y_{n-1}, y_n) \\ &\Rightarrow d(y_n, y_{n+1}) \leq \frac{\delta + kL}{1 - kL} d(y_{n-1}, y_n) \end{aligned}$$

Let $\frac{\delta + kL}{1 - kL} = k_2$, since $\delta k + kL(1 + k) < 1$,
 $k_2 < \frac{1}{k}$ and $d(y_n, y_{n+1}) \leq k_2 d(y_{n-1}, y_n)$

Case: 3 If

$$\begin{aligned} \max\{d(y_{n-1}, y_n), d(y_n, y_{n+1})\} &= d(y_n, y_{n+1}) \\ \min\{d(y_{n-1}, y_n), d(y_n, y_{n+1})\} &= d(y_{n-1}, y_n) \end{aligned}$$

Then

$$\begin{aligned} d(y_n, y_{n+1}) &\leq \delta d(y_n, y_{n+1}) + L d(y_{n-1}, y_n) \\ &\Rightarrow (1 - \delta)d(y_n, y_{n+1}) \leq L d(y_{n-1}, y_n) \\ &\Rightarrow d(y_n, y_{n+1}) \leq \frac{L}{1 - \delta} d(y_{n-1}, y_n) \end{aligned}$$

Let $\frac{L}{1 - \delta} = k_3$, since $\delta k + kL(1 + k) < 1$,
 $k_3 < 1$ and $d(y_n, y_{n+1}) \leq k_3 d(y_{n-1}, y_n)$.

Case: 4 If

$$\begin{aligned} \max\{d(y_{n-1}, y_n), d(y_n, y_{n+1})\} &= d(y_n, y_{n+1}) \\ \min\{d(y_{n-1}, y_n), d(y_n, y_{n+1})\} &= d(y_{n-1}, y_{n+1}) \end{aligned}$$

Then

$$\begin{aligned} d(y_n, y_{n+1}) &\leq \delta d(y_n, y_{n+1}) + Ld(y_{n-1}, y_{n+1}) \\ &\leq \delta d(y_n, y_{n+1}) + Lkd(y_{n-1}, y_n) + Lkd(y_n, y_{n+1}) \\ &\Rightarrow (1 - \delta - Lk)d(y_n, y_{n+1}) \leq Lkd(y_{n-1}, y_n) \\ &\Rightarrow d(y_n, y_{n+1}) \leq \frac{kL}{1 - \delta - kL} d(y_{n-1}, y_n) \end{aligned}$$

Let $\frac{kL}{1 - \delta - kL} = k_4$, since $\delta k + kL(1 + k) < 1$,
 $k_4 < \frac{1}{k}$ and $d(y_n, y_{n+1}) \leq k_4 d(y_{n-1}, y_n)$
 Let $h = \max\{k_1, k_2, k_3, k_4\}$ then $h < \frac{1}{k}$ and
 $d(y_{n+1}, y_n) \leq hd(y_n, y_{n-1}) \leq h^n d(y_0, y_1)$.

Now for $n > m$

$$\begin{aligned} d(y_m, y_n) &\leq kd(y_m, y_{m+1}) + kd(y_{m+1}, y_n) \\ &\leq kd(y_m, y_{m+1}) + k^2 d(y_{m+1}, y_{m+2}) + k^2 d(y_{m+2}, y_n) \\ &\leq kd(y_m, y_{m+1}) \\ &\quad + k^2 d(y_{m+1}, y_{m+2}) + \dots + k^{n-m} d(y_{n-1}, y_n) \\ &\leq kh^m d(y_0, y_1) \\ &\quad + k^2 h^{m+1} d(y_0, y_1) + \dots + k^{n-m} h^{n-1} d(y_0, y_1) \\ &= h^m (k + k^2 h + k^3 h^2 + \dots + k^{n-m} h^{n-m-1}) d(y_0, y_1) \\ &= h^m k (1 + (kh) + (kh)^2 + \dots + (kh)^{n-m-1}) d(y_0, y_1) \\ &< \frac{h^m k}{1 - kh} \text{ (since } kh < 1) \\ &\rightarrow 0 \text{ as } m \rightarrow \infty \end{aligned}$$

Therefore $\{y_n\}$ is a Cauchy sequence in (X, d) , since (X, d) is complete, there exist $z \in X$ such that $\lim_{n \rightarrow \infty} y_n = z$.

Assume that $g(X)$ is closed, therefore there exist a point $u \in X$ such that $z = gu$

Now we have,

$$\begin{aligned} &d(z, Fu) \\ &\leq k d(z, y_{n+1}) + k d(y_{n+1}, Fu) \\ &= k [d(z, y_{n+1}) + d(y_{n+1}, Fu)] \\ &\leq k [d(z, y_{n+1}) \\ &+ \delta \max\{d(gu, Gx_{n+1}), d(gu, fu), d(Gx_{n+1}, fx_{n+1}), \end{aligned}$$

$$\begin{aligned} &\frac{1}{2k} [d(gu, fx_{n+1}) + d(Gx_{n+1}, Fu)] \\ &\quad + L \min\{d(gu, Fu), d(gu, fx_{n+1})\}] \\ &= k [d(z, y_{n+1}) \\ &\quad + \delta \max\{d(z, y_n), d(z, fu), d(y_n, y_{n+1}), \\ &\quad \frac{1}{2k} [d(z, y_{n+1}) + d(y_n, Fu)] \\ &\quad + L \min\{d(z, Fu), d(z, y_{n+1})\}] \\ &\leq k d(z, y_{n+1}) \\ &\quad + \delta k \max\{d(z, y_n), d(z, fu), [k d(y_n, z) \\ &\quad + k d(z, y_{n+1})], \\ &\quad \frac{1}{2k} [d(z, y_{n+1}) + k d(y_n, z) + k d(z, Fu)] \\ &\quad + L \min\{d(z, Fu), d(z, y_{n+1})\} \end{aligned}$$

Taking limit as $n \rightarrow \infty$ we get,

$$d(z, fu) \leq \delta kd(z, Fu) \text{ [by the given condition}$$

$\delta k < 1]$

$$\text{Therefore, } d(z, Fu) = 0 \Rightarrow z = Fu$$

Since F and g are weakly compatible, we obtain

that,

$$gFu = Fgu$$

$$\Rightarrow gz = Fz$$

Since $F(X) \subseteq G(X)$, there exist $v \in X$ such that

$$Gv = z$$

Applying the given condition we get,

$$\begin{aligned} &d(z, fv) = d(Fu, fv) \\ &\leq \delta \max\{d(gu, Gv), d(gu, Fu), d(Gv, fv), \\ &\quad \frac{1}{2k} [d(gu, fv) + d(Gv, Fu)] \\ &\quad + L \min\{d(gu, Fu), d(gu, fv)\} \\ &\quad \therefore d(z, fv) \leq \delta d(z, fv) \\ &\Rightarrow d(z, fv) = 0 \text{ } (\because \delta k \leq 1) \\ &\quad \therefore z = fv = Gv \end{aligned}$$

Since G and f are weakly compatible, we obtain that

$$\begin{aligned} &fGv = Gfv \\ &\Rightarrow fz = Gz \end{aligned}$$

$$d(Fz, z) = d(Fz, fv)$$

$$\begin{aligned} &\leq \delta \max\{d(gz, Gv), d(gz, Fz), d(Gv, fv), \frac{1}{2k} [d(gz, fv) \\ &\quad + d(Gv, Fz)] \\ &\quad + L \min\{d(gz, Fz), d(Gv, Fz)\} \\ &= \delta \max\{d(Fz, z), d(Fz, Fz), d(z, z), \frac{1}{2k} [d(Fz, z) \\ &\quad + d(z, Fz)] \\ &\quad + L \min\{d(Fz, Fz), d(Fz, z)\} \\ &= \delta d(Fz, z) \\ &\Rightarrow d(Fz, z) = 0 \end{aligned}$$

So, $gz = Fz = z$. Similarly we get,

$$d(z, fz) = d(Fz, fz)$$

$$\begin{aligned} &\leq \delta \max\{d(gz, Gz), d(gz, Fz), d(Gz, fz), \frac{1}{2k} [d(gz, fz) \\ &\quad + d(Gz, Fz)] \\ &\quad + L \min\{d(gz, Fz), d(gz, fz)\} \\ &= \delta \max\{d(z, fz), d(z, z), d(fz, fz), \frac{1}{2k} [d(z, fz) \\ &\quad + d(fz, z)] \\ &\quad + L \min\{d(z, z), d(z, fz)\} \\ &= \delta d(z, fz) \\ &\Rightarrow d(z, fz) = 0 \end{aligned}$$

So, $Gz = fz = z$ and therefore z is common fixed point of f, g, F and G .

Uniqueness of such common fixed point:

Let $p \in X$ be also a common fixed point of f, g, F and G . Again applying the given condition we get,

$$\begin{aligned}
 d(z, p) &= d(Fz, fp) \\
 &\leq \delta \max\{d(gz, Gp), d(gz, Fz), d(Gp, fp), \frac{1}{2k} [d(gz, fp) \\
 &\quad + d(Gp, Fz)]\} \\
 &\quad + L \min\{d(gz, Fz), d(gz, fp)\} \\
 &= \delta \max\{d(z, p), d(z, z), d(p, p), \frac{1}{2k} [d(z, p) + d(p, z)]\} \\
 &\quad + L \min\{d(z, z), d(z, p)\} \\
 &= \delta d(z, p)
 \end{aligned}$$

i.e. $d(z, p) \leq \delta d(z, p)$

This implies that $d(z, p) = 0$ and so $z = p$

Hence f, g, F and G have a unique common fixed point in X .

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