Interactive TOPSIS Algorithm for Fuzzy Large Scale Two-Level Linear Multiple Objective Programming Problems

Tarek H. M. Abou-El-Enien, Mahmoud A. Abo-Sinna

***Abstract*— In this paper, we extend TOPSIS (Technique for Order Preference by Similarity Ideal Solution) for solving Large Scale Two-level Linear Multiple Objective Programming problems with fuzzy parameters in the right-hand side of the independent constraints (FLS-TL-LMOP). In order to obtain a compromise (satisfactory) solution to the above problems using the TOPSIS approach, a modified formulas for the distance function from the positive ideal solution (PIS) and the distance function from the negative ideal solution (NIS) are proposed and modeled to include all the objective functions of both the first and the second levels. An interactive decision making algorithm for generating a compromise (satisfactory) solution through TOPSIS approach is provided where the first level decision maker (FLDM) is asked to specify the membership function for each fuzzy parameter , the maximum negative and positive tolerance values, the power p of the distance functions, the degree α and the relative importance of the objectives. Finally, a numerical example is given to clarify the main results developed in the paper.**

***Index Terms*— Interactive decision making, multiple objective programming problems, two-level programming, TOPSIS, block angular structure, large scal programming, fuzzy programming, fuzzy parameters.**

I. Introduction

The decentralized planning has been recognized as an important decision-making problem. It seeks to find a simultaneous compromise among the various objective functions of the different divisions. Twoi-level programming, a tool for modeling decentralized decisions, consists of the objective(s) of the leader at its first level and that is of the follower at the second level. The decision-maker at each level attempts to optimize his individual objective, which usually depends in part on the variables controlled by the decision-maker at the other levels and their final decisions are executed sequentially where the upper-level decision-maker makes his decision firstly, [4, 5, 25].

Abo–Sinna, M. A. and Abou-El-Enien, T. H. M. [1], introduce an algorithm for solving large scale multiple objective decision making (LSMODM) problems by use of TOPSIS.Abo–Sinna, M. A. and Abou-El-Enien, T. H. M. [2], extend TOPSIS for solving interactive large scale multiple Objective programming problems involving fuzzy parameters. An interactive fuzzy decision making algorithm for generating α-Pareto optimal solution through TOPSIS approach is provided where the decision maker (DM) is asked to specify the degree α and the relative importance of objectives.

Abou-El-Enien, T. H. M. [6], focus on the solution of a Large Scale Integer Linear Vector Optimization Problems with chance constraints (CHLSILVOP) using TOPSIS approach, where such problems has block angular structure of the constraints. He introduced an algorithm based on TOPSIS approach to solve CHLSILVOP with constraints of block angular structure.

Abou-El-Enien, T. H. M. [8] ,extend TOPSIS method for solving Linear Fractional Vector Optimization problems (LFVOP) of a special type, An interactive decision making algorithm for generating a compromise solution through TOPSIS approach is provided where the decision maker (DM) is asked to specify the degree α and the relative importance of the objectives. Finally, a numerical example is given to clarify the main results developed in the paper.

Baky, I. A. and Abo-Sinna, M. A. [12] proposed a TOPSIS algorithm for bi-level multiple objective decision making problems.

Abo–Sinna, M. A. and Abou-El-Enien, T. H. M. [3] , extend TOPSIS method for solving Large Scale Bi-level Linear Vector Optimization Problems (LS-BL-LVOP). They further extended the concept of TOPSIS for LS-BL-LVOP.

Abou-El-Enien, T. H. M. [9], extended TOPSIS for solving a special type of Two-Level Integer Linear Multiple Objectives Decision Making (ST-TL-IL MODM) Problems with block angular structure. In order to obtain a compromise (satisfactory) integer solution to the ST-TL-ILMODM problems with block angular structure using the proposed TOPSIS approach, a modified formulas for the distance function from the positive ideal solution (PIS) and the distance function from the negative ideal solution (NIS) are proposed and modeled to include all the objective functions of the two levels. In every level, as the measure of "Closeness" dp-metric is used, a k-dimensional objective space is reduced to two –dimensional objective space by a first-order compromise procedure.

Abou-El-Enien, T. H. M. [10], extended TOPSIS method for solving Two-Level Large Scale Linear Multiobjective Optimization Problems with Stochastic Parameters in the right hand side of the constraints (TL-LSLMOP-SP) rhs of block angular structure. In order to obtain a compromise (satisfactory) solution to the (TL-LSLMOP-SP) rhs of block angular structure using the proposed TOPSIS method, a modified formulas for the distance function from the positive ideal solution (PIS) and the distance function from the negative ideal solution (NIS) are proposed and modeled to include all the objective functions of the two levels.

Abou-El-Enien, T. H. M. [7] , presents many algorithms for solving different kinds of Large Scale Linear

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Multiobjective Optimization (LSLMO) problems using TOPSIS method.

Abou-El-Enien, T. H. M. & El-Feky, Sh. F. [11], present a review on theory, applications and softwares of bi-level, multi-level multiple criteria decision making and TOPSIS approach.

In this paper, we extend TOPSIS for solving FLS-TL-LMOP problems. Also, we extended the concept of TOPSIS [Lia *et al.* (20)] for FLS-TL- LMOP problems.

In the following section, we will give the formulation of Large Scale Two-level Linear Multiple Objective Programming problems (LS-TL-LMOP) with fuzzy parameters in the right-hand side of the independent constraints (FLS-TL-LMOP). The TOPSIS approach is presented in section (3). By use of TOPSIS, we will propose an interactive algorithm for solving α- LS-TL-LMOP problems in section (4), where the FLDM is asked to specify the membership function for each fuzzy parameter, the maximum negative and positive tolerance values, the power p of the distance functions, [7, 23], the degree α and the relative importance of the objectives. We will also give a numerical example in section (5) for the sake of illustration. Finally, concluding remarks will be given in section (6).

II. Formulation of the Problem

Consider there are two levels in a hierarchy structure with a first - level decision maker (FLDM) and a second - level decision maker (SLDM). Let the FLS-TL-LMOP problem with fuzzy parameters in the right-hand side of the independent constraints which has a block angular structure as follows :

[*FLDM*]

$$\underset{X_{I_1}}{Maximize/Minimize}\ Z_{I_1}(X_{I_1},X_{I_2}) = \underset{X_{I_1}}{Maximize/Minimize}\left(z_{I_1 1}(X_{I_1},X_{I_2}),\ldots,z_{I_1 k_{I_1}}(X_{I_1},X_{I_2})\right)$$

[*SLDM*]

$$\underset{X_{I_2}}{Maximize/Minimize}\ Z_{I_2}(X_{I_1},X_{I_2}) = \underset{X_{I_2}}{Maximize/Minimize}\left(z_{I_2 1}(X_{I_1},X_{I_2}),\ldots,z_{I_2 k_{I_2}}(X_{I_1},X_{I_2})\right)$$

subject to (1)

$$X \in M = \{X \in R^n: \sum_{j=1}^{q} A_j X_j \le b_0,\quad D_j X_j \le \tilde{y}_j \quad X_j \ge 0, j = 1,2,\ldots,q, q > 1\}$$

k : the number of objective functions,
k_{I_1} : the number of objective functions of the FLDM
k_{I_2} : the number of objective functions of the SLDM
n_{I_1} : the number of variables of the FLDM
n_{I_2} : the number of variables of the SLDM
q : the number of subproblems,
m : the number of constraints,
n : the numer of variables,
n_j : the number of variables of the j^{th} subproblem, $j=1,2,\ldots,q$,
m_o : the number of the common constraints represented by $\sum_{j=1}^{q} A_j X_j \le b_0$,
m_j : the number of independent constraints of the j^{th} subproblem represented by $D_j X_j \le \tilde{y}_j$, $j=1,2,\ldots,q$.
A_j : an $(m_o \times n_j)$ coefficient matrix,
D_j : an $(m_j \times n_j)$ coefficient matrix,
b_o : an m_o-dimensional column vector of right-hand sides of the common constraints whose elements are onstants,
$\tilde{y}_j$: an m_j-dimensional column vector of independent constraints right-hand sides whose elements are fuzzy parameters for the j^{th} subproblem, $j=1,2,\ldots,q$.
C_{ij} : an n_j-dimensional row vector for the j^{th} subproblem in the i^{th} objective function,
R : the set of all real numbers,
X : an n-dimensional column vector of variables,
X_j : an n_j -dimensional column vector of variables for the j^{th} subproblem, $j=1,2,\ldots,q$,
X_{I_1} : an n_{I_1} - dimensional column vector of variables of the FLDM,
X_{I_2} : an n_{I_2} - dimensional column vector of variables of the SLDM,
$K = \{1,2,\ldots,k\}$
$N = \{1,2,\ldots,n\}$,
$R^n = \{X=(x_1, x_2,\ldots,x_n)^T : x_i \in R, i \in N\}$.

If the objective functions are linear, then the objective function can be written as follows:

$$z_i(X) = \sum_{j=1}^{q} z_{ij} = \sum_{j=1}^{q} C_{ij} X_{ij}, i = 1,2,\ldots,k. \qquad (2)$$

Where,
$\sum_{j=1}^{q} z_{tj}(X)$: Objective Function for Maximization, $t \in K_1 \subset K$,
$\sum_{j=1}^{q} z_{vj}(X)$: Objective Function for Minimization, $v \in K_2 \subset K$.

Throughout this paper, we assume that the column vectors of fuzzy parameters $\tilde{y}_j, j=0,1,2,\ldots,q$, are characterized as the column vectors of fuzzy numbers, [18, 25, 26].

It is appropriate to recall that a real fuzzy number $\tilde{y}$ whose membership function $\mu_{\tilde{y}}(y)$ is defined as, [18, 25, 26].

(1) A continuous mapping from R^1 to the closed interval $[0,1]$,

(2) $\mu_{\tilde{y}}(y)=0$ for all $y \in (-\infty, y_1]$,

(3) Strictly increasing on $[y_1,y_2]$,

(4) $\mu_{\tilde{y}}(y)=1$ for all $y \in [y_2,y_3]$,

(5) Strictly decreasing on $[y_3,y_4]$,

(6) $\mu_{\tilde{y}}(y)=0$ for all $y \in [y_4,+\infty)$.

We first introduce the concept of α-level set or α-cut of the fuzzy numbers $\tilde{y}_j, j=1,2,\ldots,q$, as follows, [18, 25, 26].

Definition 1. The α-level set of the fuzzy numbers $\tilde{y}_j$, $j=0,1,2,\ldots,q$, as the ordinary set $(\tilde{y}_j)_\alpha$ for which the degree of their membership function exceeds the level $\alpha \in [0,1]$:

$$\left(\tilde{y}_j\right)_\alpha = \left\{y: \mu_{\tilde{y}_{j,s}}\left(y_{j,s}\right) \geq \alpha,\ j = 1,2,\dots,q, s = 1,2,\dots,m_j\right\} \tag{3}$$

For a certain degree α, the (LS-TL-LMOP) problem (1) can be understood as the following nonfuzzy α- Large Scale Ti-level Linear Multiple Objective Programming (α- LS-TL-LMOP) problem, [19, 21, 26]:

[FLDM]

$$\underset{X_{I_1}}{Maximize/Minimize}\ Z_{I_1}(X_{I_1}, X_{I_2}) = \underset{X_{I_1}}{Maximize/Minimize}\left(z_{I_1 1}(X_{I_1}, X_{I_2}), \dots, z_{I_1 k_{I_1}}(X_{I_1}, X_{I_2})\right) \tag{4-a}$$

[SLDM]

$$\underset{X_{I_2}}{Maximize/Minimize}\ Z_{I_2}(X_{I_1}, X_{I_2}) = \underset{X_{I_2}}{Maximize/Minimize}\left(z_{I_2 1}(X_{I_1}, X_{I_2}), \dots, z_{I_2 k_{I_2}}(X_{I_1}, X_{I_2})\right) \tag{4-b}$$

subject to

$$X \in M^{/} = \{X \in R^n: \textstyle\sum_{j=1}^{q} A_j X_j \leq b_0, \tag{4-c}$$
$$D_j X_j \leq y_j, \tag{4-d}$$
$$X_j \geq 0, j = 1,2,\dots,q, q > 1, \tag{4-e}$$
$$y_j \in \left(\tilde{y}_j\right)_\alpha \ \} \tag{4-f}$$

$$L_j \leq y_j \leq U_j, j = 1,2,\dots,q, q > 1\} \tag{4-f}$$

In the (α- LS-TL-MODM) problem (4), the parameters y_j, $j=0, 1,\dots,q$, are treated as decision variables rather than constants.

Thus, the (α- LS-TL-MODM) problem (4) can be written as follows, [19, 23,28]:

[α-FLDM]

$$\underset{X_{I_1}}{Maximize/Minimize}\ Z_{I_1}(X_{I_1}, X_{I_2}) = \underset{X_{I_1}}{Maximize/Minimize}\left(z_{I_1 1}(X_{I_1}, X_{I_2}), \dots, z_{I_1 k_{I_1}}(X_{I_1}, X_{I_2})\right) \tag{5-a}$$

[α-SLDM]

$$\underset{X_{I_2}}{Maximize/Minimize}\ Z_{I_2}(X_{I_1}, X_{I_2}) = \underset{X_{I_2}}{Maximize/Minimize}\left(z_{I_2 1}(X_{I_1}, X_{I_2}), \dots, z_{I_2 k_{I_2}}(X_{I_1}, X_{I_2})\right) \tag{5-b}$$

subject to

$$X \in M^{//} = \{X \in R^n: \textstyle\sum_{j=1}^{q} A_j X_j \leq b_0, \tag{5-c}$$
$$D_j X_j \leq y_j, \tag{5-d}$$
$$X_j \geq 0, j = 1,2,\dots,q, q > 1, \tag{5-e}$$
$$L_j \leq y_j \leq U_j, j = 1,2,\dots,q, q > 1\} \tag{5-f}$$

It should be noted that the constraint (4-f) is replaced by the equivalent constraint (5-f), where L_j and U_j are lower and upper bound on y_j, respectively.

Problem (5) can be written as follows:

[α-FLDM]

$$\underset{X_{I_1}}{Maximize/Minimize}\ Z_{I_1}(X_{I_1}, X_{I_2}) = \underset{X_{I_1}}{Maximize/Minimize}\left(z_{I_1 1}(X_{I_1}, X_{I_2}), \dots, z_{I_1 k_{I_1}}(X_{I_1}, X_{I_2})\right) \tag{6-a}$$

[α-SLDM]

$$\underset{X_{I_2}}{Maximize/Minimize}\ Z_{I_2}(X_{I_1}, X_{I_2}) = \underset{X_{I_2}}{Maximize/Minimize}\left(z_{I_2 1}(X_{I_1}, X_{I_2}), \dots, z_{I_2 k_{I_2}}(X_{I_1}, X_{I_2})\right) \tag{6-b}$$

subject to

$$X^{/} \in M^{//} = \{X^{/} \in R^{n+(m-m_o)}: \textstyle\sum_{j=1}^{q} A_j^{/} X_j^{/} \leq b_o^{/}, \tag{6-c}$$
$$D_j^{/} X_j^{/} \leq b_j^{/}, \tag{6-d}$$
$$X_j^{/} \geq 0, j = 1,2,\dots,q, q > 1\} \tag{6-e}$$

where

$X_j^{/} = (X_j, y_j)^T$ is an $(n_j + m_j)$ -dimensional column vector of variables for the j^{th} subproblem, $j=1,2,\dots,q,$

$A_j^{/} = (A_j \,|\, O)$ is an $(m_o \times (n_j + m_j))$ matrix , where O is an $(m_o \times m_j)$ zero matrix, $D_j^{/}$ is a $(3m_j \times (n_j + m_j))$ matrix which is the coefficient of the left-hand side of the following system:

$$D_j X_j - y_j \leq 0 \tag{7-a}$$
$$y_j \leq U_j \tag{7-b}$$
$$y_j \geq L_j,\ j = 1,2,\dots,q, q > 1 \tag{7-c}$$

and $b_j^{/}$ is a $3m_j$-dimensional column vector of independent constraints right-hand side of system (7) for the j^{th} subproblem, $j=1,2,\dots,q$.

III. TOPSIS for α- LS-TL-LMOP Problems

This section present the two phases of TOPSIS approach for α- LS-TL-LMOP Problems.

3-1. Phase (I):

Consider the *α-FLDM* problem of the α-LS-TL-LMOP Problem (7):

[*α-FLDM*]

$$\underset{X_{I_1}}{Maximize/Minimize}\ Z_{I_1}(X_{I_1}, X_{I_2}) = \underset{X_{I_1}}{Maximize/Minimize}\left(z_{I_1 1}(X_{I_1}, X_{I_2}), \dots, z_{I_1 k_{I_1}}(X_{I_1}, X_{I_2})\right)$$

subject to (8)

$X^{/} \in M^{//}$.

Observing that the α-FLDM problem (8) involves fuzzy parameters in the right-hand side of the independent constraints, it is evident that the notion of Pareto optimality, [28], defined for the FLDM problem (8) cannot be applied directly. Thus, it seems essential to extend the notion of usual Pareto optimality in some sence. Based on the definition of α-level set of the fuzzy numbers, we introduce the concept of *α*-Pareto optimal solution to (α- FLDM) problem (8) in the following definition, [2, 12, 13]:

Definition 2. A solution X_j^*, $j=1,2,\dots,q$, is said to be an *α*-Pareto optimal solution to the (α- *FLDM*) problem (8), if and only if there does not exist another X_j, $j=1,2,\dots,q$, $y_j \in (\tilde{y}_j)_\alpha$, such that $z_{ij}(X_j) \geq z_{ij}(X_j^*)$, $i=1,2,\dots,k$, with strictly inequality holding for at least one *i*, where the corresponding value of the parameter y^* is called α-level optimal parameter.

In order to use the distance family of equation (6) to resolve problem (8), we must first find PIS(f^*) and NIS(f^-) which are [20]:

$$z^{*\alpha-\text{FLDM}} = \underset{X^{/} \in M^{//}}{Maximize(or\ Minimize)} \sum_{j=1}^{q} z_{tj}^{\alpha-\text{FLDM}}(X) \left(or\ \sum_{j=1}^{q} z_{vj}^{\alpha-FLDM}(X)\right), \forall t(and\ v) \quad \text{(9-a)}$$

$$z^{-\alpha-\text{FLDM}} = \underset{X^{/} \in M^{//}}{Minimize(or\ Maximize)} \sum_{j=1}^{q} z_{tj}^{\alpha-\text{FLDM}}(X) \left(or\ \sum_{j=1}^{q} z_{vj}^{\alpha-FLDM}(X)\right), \forall t(and\ v) \quad \text{(9-b)}$$

where $K = K_1 \cup K_2$, $z^{*\alpha-\text{FLDM}} = \left(z_1^{*\alpha-FLDM}, z_2^{*\alpha-FLDM}, \dots, z_{k_{I_1}}^{*\alpha-FLDM}\right)$ and $z^{-\alpha-\text{FLDM}} = (z_1^{-\alpha-FLDM}, z_2^{-\alpha-FLDM}, \dots, z_{k_{I_1}}^{-\alpha-FLDM})$ are the individual positive (negative) ideal solutions for the α-FLDM problem.

Using the PIS and the NIS for the α- FLDM problem, we obtain the following distance functions from them, respectively [7, 20, 23]:

$$d_P^{PIS^{\alpha-\text{FLDM}}} = \left(\sum_{t\in K_1} w_t^p \left(\frac{\sum_{j=1}^{q} z_{tj}^{*\alpha-\text{FLDM}} - \sum_{j=1}^{q} z_{tj}^{\alpha-\text{FLDM}}(X)}{\sum_{j=1}^{q} z_{tj}^{*\alpha-\text{FLDM}} - \sum_{j=1}^{q} z_{tj}^{-\alpha-FLDM}}\right)^p + \sum_{v\in K_2} w_v^p \left(\frac{\sum_{j=1}^{q} f_{vj}^{\alpha-FLDM}(X) - \sum_{j=1}^{q} f_{vj}^{*\alpha-FLDM}}{\sum_{j=1}^{q} z_{vj}^{-\alpha-FLDM} - \sum_{j=1}^{q} z_{vj}^{*\alpha-FLDM}}\right)^p\right)^{1/p} \quad \text{(10-a)}$$

and

$$d_P^{NIS^{\alpha-\text{FLDM}}} = \left(\sum_{t\in K_1} w_t^p \left(\frac{\sum_{j=1}^{q} z_{tj}^{\alpha-FLDM}(X) - \sum_{j=1}^{q} z_{tj}^{-\alpha-FLDM}}{\sum_{j=1}^{q} z_{tj}^{*\alpha-FLDM} - \sum_{j=1}^{q} z_{tj}^{-\alpha-FLDM}}\right)^p + \sum_{v\in K_2} w_v^p \left(\frac{\sum_{j=1}^{q} z_{vj}^{-\alpha-FLDM} - \sum_{j=1}^{q} z_{vj}^{\alpha-FLDM}(X)}{\sum_{j=1}^{q} z_{vj}^{-\alpha-FLDM} - \sum_{j=1}^{q} z_{vj}^{*\alpha-FLDM}}\right)^p\right)^{1/p} \quad \text{(10-b)}$$

where $w_i = 1,2,\dots,k$, are the relative importance (weighs) of objectives, and $p = 1,2,\dots,\infty$.

In order to obtain a compromise solution for the α-FLDM problem, we transfer the α- FLDM problem (8) into the following bi-objective problem with two commensurable (but often conflicting) objectives [20]:

$Minimize\ d_p^{PIS^{\alpha-\text{FLDM}}}(X)$

$Maximize\ d_p^{NIS^{\alpha-\text{FLDM}}}(X)$

subject to (11)

$X^{/} \in M^{//}$ where $p = 1,2,\dots,\infty$.

Since these two objectives are usually conflicting to each other, we can simultaneously obtain their individual optima. Thus, we can use membership functions to represent these individual optima. Assume that the membership functions ($\mu_1(X)$ and $\mu_2(X)$) of two objective functions are linear. Then, based on the preference concept, we assign a larger degree to the one with shorter distance from the PIS for $\mu_1(X)$ and assign a larger degree to the one with farther distance from NIS for $\mu_2(X)$. Therefore, as shown in figure (1), $\mu_1(X) \equiv \mu_{d_p^{PIS^{\alpha-\text{FLDM}}}}(X)$ and $\mu_2(X) \equiv \mu_{d_p^{NIS^{\alpha-\text{FLDM}}}}(X)$ can be obtained as the following (see [14, 25, 28]):

$$\mu_1(X) = \begin{cases} 1, & if\ d_p^{PIS^{\alpha-\text{FLDM}}}(X) < \left(d_p^{PIS^{\alpha-\text{FLDM}}}\right)^*, \\ 1 - \dfrac{d_p^{PIS^{\alpha-\text{FLDM}}}(X) - \left(d_p^{PIS^{\alpha-\text{FLDM}}}\right)^*}{\left(d_p^{PIS^{\alpha-\text{FLDM}}}\right)^{/} - \left(d_p^{PIS^{\alpha-\text{FLDM}}}\right)^*}, & if\ \left(d_p^{PIS^{\alpha-\text{FLDM}}}\right)^- \geq d_p^{PIS^{\alpha-\text{FLDM}}}(X) \geq \left(d_p^{PIS^{\alpha-\text{FLDM}}}\right)^*, \\ 0, & if\ d_p^{PIS^{\alpha-\text{FLDM}}}(X) > \left(d_p^{PIS^{\alpha-\text{FLDM}}}\right)^-, \end{cases} \quad \text{(12-a)}$$

$$\mu_2(X)=\begin{cases}1, & if\, d_p^{NIS^{\alpha-\mathrm{FLDM}}}(X)>\left(d_p^{NIS^{\alpha-\mathrm{FLDM}}}\right)^*,\\ 1-\dfrac{\left(d_p^{NIS^{\alpha-\mathrm{FLDM}}}\right)^*-d_p^{NIS^{\alpha-\mathrm{FLDM}}}(X)}{\left(d_p^{NIS^{\alpha-\mathrm{FLDM}}}\right)^*-\left(d_p^{NIS^{\alpha-\mathrm{FLDM}}}\right)^-}, & if\left(d_p^{NIS^{\alpha-\mathrm{FLDM}}}\right)^-\le d_p^{NIS^{\alpha-\mathrm{FLDM}}}(X)\le\left(d_p^{NIS^{\alpha-\mathrm{FLDM}}}\right)^*,\\ 0, & if\, d_p^{NIS^{\alpha-\mathrm{FLDM}}}(X)<\left(d_p^{NIS^{\alpha-\mathrm{FLDM}}}\right)^-,\end{cases}\quad (12\text{-b})$$

where

$\left(d_p^{PIS^{\alpha-\mathrm{FLDM}}}\right)^*=\underset{X^{/}\in M^{//}}{Minimize}\; d_p^{PIS^{\alpha-\mathrm{FLDM}}}(X)$ *and the solution is* $X^{PIS^{\alpha-\mathrm{FLDM}}}$,

$\left(d_p^{NIS^{\alpha-\mathrm{FLDM}}}\right)^*=\underset{X^{/}\in M^{//}}{Maximize}\; d_p^{NIS^{\alpha-\mathrm{FLDM}}}(X)$ *and the solution is* $X^{NIS^{\alpha-\mathrm{FLDM}}}$,

$\left(d_p^{PIS^{\alpha-\mathrm{FLDM}}}\right)^-=d_p^{PIS^{\alpha-\mathrm{FLDM}}}\left(X^{NIS^{\alpha-\mathrm{FLDM}}}\right)$ *and* $\left(d_p^{NIS^{\alpha-\mathrm{FLDM}}}\right)^-=d_p^{NIS^{\alpha-\mathrm{FLDM}}}\left(X^{PIS^{\alpha-\mathrm{FLDM}}}\right)$

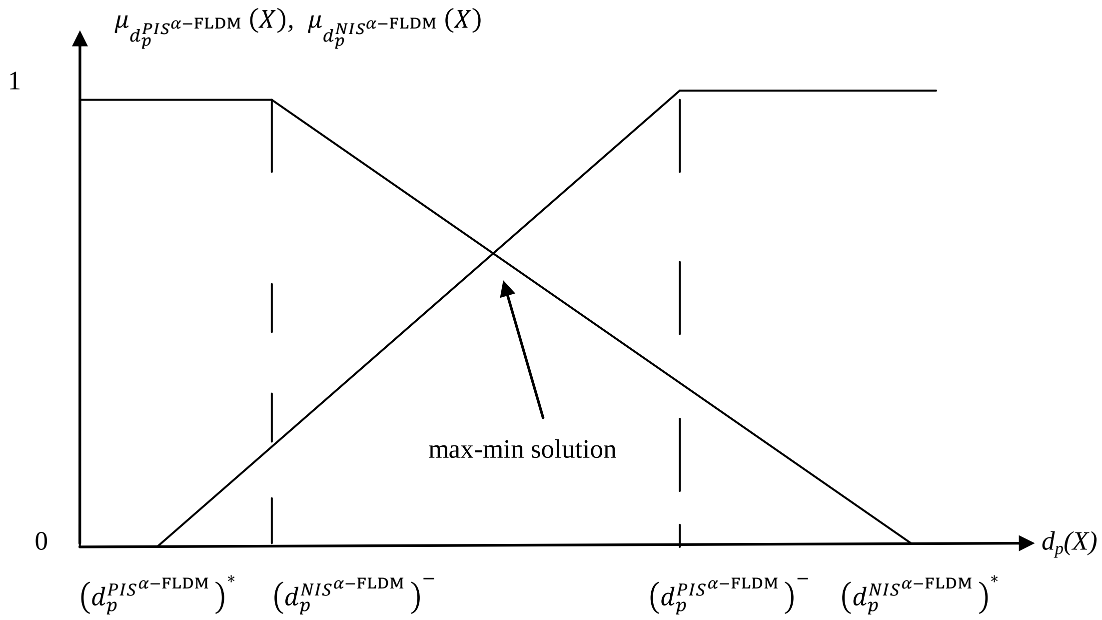

Figure (1): The membership functions of $\mu_{d_p^{PIS^{\alpha-\mathrm{FLDM}}}}(X)$ *and* $\mu_{d_p^{NIS^{\alpha-\mathrm{FLDM}}}}(X)$

Now, by applying the max-min decision model which is proposed by R. E. Bellman and L. A. Zadeh [14] and extended by H. –J. Zimmermann [28], we can resolve problem (10). The satisfying decision of the α-FLDM Problem of the α-LS-TL-LMOP Problem, $X^{/*^{\alpha-\mathrm{FLDM}}}=(X_{I_1}^{/*^{\alpha-\mathrm{FLDM}}},X_{I_2}^{/*^{\alpha-\mathrm{FLDM}}})$, may be obtained by solving the following model:

$$\mu_D\left(X^{/*^{\alpha-\mathrm{FLDM}}}\right)=\underset{X^{/}\in M^{//}}{Maximize}\;\{Min.\left(\mu_1(X),\ \mu_2(X)\right)\}\qquad(13)$$

Finally, if $\delta^{\alpha-\mathrm{FLDM}}=Minimize\,(\mu_1(X),\ \mu_2(X))$, the model (13) is equivalent to the form of Tchebycheff model (see [17]), which is equivalent to the following model:

$$Maximize\ \delta^{\alpha-FLDM},\qquad(14\text{-a})$$

subject to

$$\mu_1(X)\ge\delta^{\alpha-FLDM},\qquad(14\text{-b})$$

$$\mu_2(X)\ge\delta^{\alpha-FLDM},\qquad(14\text{-c})$$

$$X^{/}\in M^{//},\quad \delta^{\alpha-FLDM}\in[0,1],\qquad(14\text{-d})$$

where $\delta^{\alpha-FLDM}$ is the satisfactory level for both criteria of the shortest distance from the PIS and the farthest distance from the *NIS*. It is well known that if the Pareto optimal solution of (8) is the vector $\left(\delta^{*^{\alpha-FLDM}},X^{/*^{\alpha-FLDM}}\right)$, then $X^{/*^{\alpha-FLDM}}$ is a nondominated solution of (10) and a satisfactory solution of the α- FLDM problem.

The basic concept of the bi-level programming technique is that the FLDM sets his/her goals and/or decisions with possible tolerances which are described by membership functions of fuzzy set theory. According to this concept, let τ_i^L and $\tau_i^R, i=1,2,\dots,n_{I_1}$ be the maximum acceptable negative and positive tolerance (relaxation) values on the decision vector considered by the FLDM, $X_{I_1}^{/*^{\alpha-FLDM}}=\left(x_{I_11}^{/*^{\alpha-FLDM}},x_{I_12}^{/*^{\alpha-FLDM}},\dots,x_{I_1n_{I_1}}^{/*^{\alpha-FLDM}}\right)$. The tolerances give the SLDM an extent feasible region to search for the satisfactory solution. If the feasible region is empty, the negative and positive tolerances must be increased to give the SLDM an extent feasible region to search for the satisfactory solution, [5, 24, 27]. The linear membership functions (Figure 2) for each of the n_{I_1} components of the decision vector $\left(x^{/*\alpha-FLDM}_{I_1 1}, x^{/*\alpha-FLDM}_{I_1 2}, \ldots, x^{/*\alpha-FLDM}_{I_1 n_{I_1}}\right)$ controlled by the FLDM can be formulated as:

$$\mu_{I_1 i}(x^{/}_{I_1 i}) = \begin{cases} \dfrac{x^{/}_{I_1 i} - \left(x^{/*\alpha-FLDM}_{I_1 i} - \tau_i^L\right)}{\tau_i^L} & if\ x^{/*\alpha-FLDM}_{I_1 i} - \tau_i^L \le x^{/}_{I_1 i} \le x^{/*\alpha-FLDM}_{I_1 i} \\ \dfrac{\left(x^{/*\alpha-FLDM}_{I_1 i} + \tau_i^R\right) - x^{/}_{I_1 i}}{\tau_i^R} & if\ x^{/*\alpha-FLDM}_{I_1 i} \le x^{/}_{I_1 i} \le x^{/*\alpha-FLDM}_{I_1 i} + \tau_i^R,\ i = 1,2,\ldots,n_{I_1} + m_j, \\ 0 & if\ otherwise, \end{cases} \tag{15}$$

It may be noted that, the decision maker may desire to shift the range of $x^{/}_{I_1 i}$. Following Pramanik & Roy [24] and Sinha [27], this shift can be achieved.

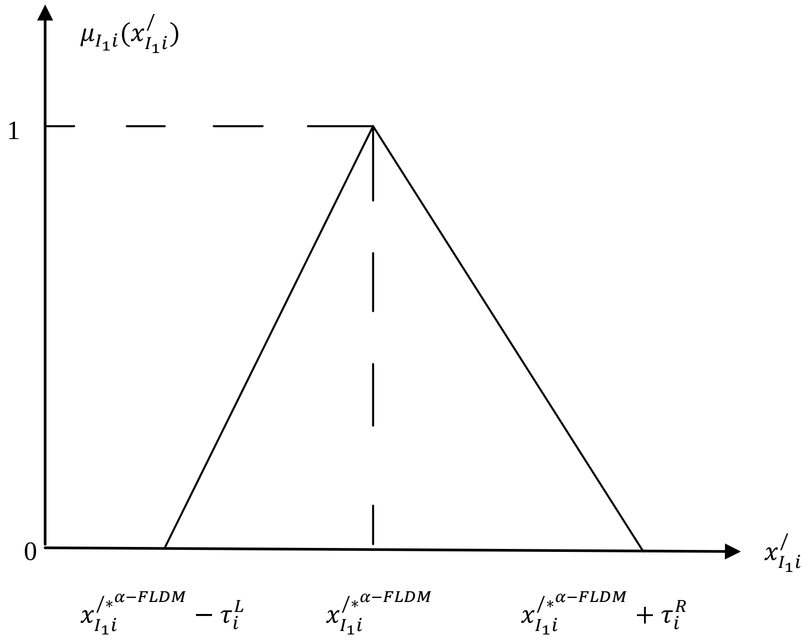

Figure (2): The membership function of the decision variable $x^{/}_{I_1 i}$

4-2. Phase (II):

The α-SLDM problem can be written as follows:

[α-SLDM]

$$\underset{X_{I_2}}{Maximize/Minimize}\ Z_{I_2}(X_{I_1}, X_{I_2}) = \underset{X_{I_2}}{Maximize/Minimize}\left(z_{I_2 1}(X_{I_1}, X_{I_2}), \ldots, z_{I_2 k_{I_2}}(X_{I_1}, X_{I_2})\right)$$

$$subject\ to \tag{16}$$

$$X^{/} \in M^{//}$$

In order to use the distance family of equation (10) to resolve problem (9), we must first find $PIS(z^*)$ and $NIS(z^-)$ which are [7, 20]:

$$f^{*\alpha-\text{SLDM}} = \underset{X_{I_2}}{Maximize(or\ Minimize)} \sum_{j=1}^{q} f_{tj}(X) \left(or\ \sum_{j=1}^{q} f_{vj}(X)\right), \forall t(and\ v) \tag{17-a}$$

$$f^{-\alpha-\text{SLDM}} = \underset{X_{I_2}}{Minimize(or\ Maximize)} \sum_{j=1}^{q} f_{tj}(X) \left(or\ \sum_{j=1}^{q} f_{vj}(X) \right), \forall t (and\ v) \quad (17\text{-b})$$

where $K = K_1 \cup K_2$,

$f^{*\alpha-\text{SLDM}} = \left(f_1^{*\alpha-\text{SLDM}}, f_2^{*\alpha-\text{SLDM}}, \ldots, f_{k_{I_2}}^{*\alpha-\text{SLDM}} \right)$ and $f^{-\alpha-\text{SLDM}} = (f_1^{-\alpha-\text{SLDM}}, f_2^{-\alpha-\text{SLDM}}, \ldots, f_{k_{I_2}}^{-\alpha-\text{SLDM}})$ are the individual positive (negative) ideal solutions for the α-SLDM problem.

In order to obtain a compromise (satisfactory) solution to the α-LS-TL-LMOP problems using TOPSIS approach, the distance family of (18) to represent the distance function from the positive ideal solution, $d_P^{PIS^{\alpha-\text{TL}}}$, and the distance function from the negative ideal solution, $d_P^{NIS^{\alpha-\text{TL}}}$, can be proposed, in this paper, for the objectives of the α-FLDM problem and the α-SLDM problem as follows:

$$d_P^{PIS^{\alpha-\text{TL}}} = \left(\sum_{t\in K_1} w_t^p \left(\frac{\sum_{j=1}^{q} z_{tj}^{*\alpha-\text{FLDM}} - \sum_{j=1}^{q} z_{tj}^{\alpha-\text{FLDM}}(X)}{\sum_{j=1}^{q} z_{tj}^{*\alpha-\text{FLDM}} - \sum_{j=1}^{q} z_{tj}^{-\alpha-FLDM}} \right)^p + \sum_{v\in K_2} w_v^p \left(\frac{\sum_{j=1}^{q} z_{vj}^{\alpha-FLDM}(X) - \sum_{j=1}^{q} z_{vj}^{*\alpha-FLDM}}{\sum_{j=1}^{q} z_{vj}^{-\alpha-FLDM} - \sum_{j=1}^{q} z_{vj}^{*\alpha-FLDM}} \right)^p + \sum_{t\in K_1} w_t^p \left(\frac{\sum_{j=1}^{q} z_{tj}^{*\alpha-SLDM} - \sum_{j=1}^{q} z_{tj}^{\alpha-SLDM}(X)}{\sum_{j=1}^{q} z_{tj}^{*\alpha-SLDM} - \sum_{j=1}^{q} z_{tj}^{-\alpha-SLDM}} \right)^p + \sum_{v\in K_2} w_v^p \left(\frac{\sum_{j=1}^{q} z_{vj}^{\alpha-SLDM}(X) - \sum_{j=1}^{q} z_{vj}^{*\alpha-SLDM}}{\sum_{j=1}^{q} z_{vj}^{-\alpha-SLDM} - \sum_{j=1}^{q} z_{vj}^{*\alpha-SLDM}} \right)^p \right)^{1/p} \quad (18\text{-a})$$

and

$$d_P^{NIS^{\alpha-\text{TL}}} = \left(\sum_{t\in K_1} w_t^p \left(\frac{\sum_{j=1}^{q} z_{tj}^{\alpha-FLDM}(X) - \sum_{j=1}^{q} z_{tj}^{-\alpha-FLDM}}{\sum_{j=1}^{q} z_{tj}^{*\alpha-FLDM} - \sum_{j=1}^{q} z_{tj}^{-\alpha-FLDM}} \right)^p + \sum_{v\in K_2} w_v^p \left(\frac{\sum_{j=1}^{q} z_{vj}^{-\alpha-FLDM} - \sum_{j=1}^{q} z_{vj}^{\alpha-FLDM}(X)}{\sum_{j=1}^{q} z_{vj}^{-\alpha-FLDM} - \sum_{j=1}^{q} z_{vj}^{*\alpha-FLDM}} \right)^p + \sum_{t\in K_1} w_t^p \left(\frac{\sum_{j=1}^{q} z_{tj}^{\alpha-SLDM}(X) - \sum_{j=1}^{q} z_{tj}^{-\alpha-SLDM}}{\sum_{j=1}^{q} z_{tj}^{*\alpha-SLDM} - \sum_{j=1}^{q} z_{tj}^{-\alpha-SLDM}} \right)^p + \sum_{v\in K_2} w_v^p \left(\frac{\sum_{j=1}^{q} z_{vj}^{-\alpha-SLDM} - \sum_{j=1}^{q} z_{vj}^{\alpha-SLDM}(X)}{\sum_{j=1}^{q} z_{vj}^{-\alpha-SLDM} - \sum_{j=1}^{q} z_{vj}^{*\alpha-SLDM}} \right)^p \right)^{1/p} \quad (18\text{-b})$$

where $w_i = 1,2, \ldots, k$, are the relative importance (weighs) of objectives, and $p = 1,2, \ldots, \infty$.

In order to obtain a compromise solution, we transfer problem (18) into the following two-objective problem with two commensurable (but often conflicting) objectives [7, 20]:

$$\begin{aligned} & Minimize\ d_p^{PIS^{\alpha-\text{TL}}}(X) \\ & Maximize\ d_p^{NIS^{\alpha-\text{TL}}}(X) \\ & \text{subject to} \\ & X^{/} \in M^{//} \end{aligned} \quad (19)$$

where $p = 1,2, \ldots, \infty$.

Since these two objectives are usually conflicting to each other, we can simultaneously obtain their individual optima. Thus, we can use membership functions to represent these individual optima. Assume that the membership functions ($\mu_3(X)$ and $\mu_4(X)$) of two objective functions are linear. Then, based on the preference concept, we assign a larger degree to the one with shorter distance from the PIS for $\mu_3(X)$ and assign a larger degree to the one with farther distance from NIS for $\mu_4(X)$. Therefore, as shown in figure (3), $\mu_3(X) \equiv \mu_{d_p^{PIS^{\alpha-\text{TL}}}}(X)$ and $\mu_4(X) \equiv \mu_{d_p^{NIS^{\alpha-\text{TL}}}}(X)$ can be obtained as the following (see [14, 25, 28]):

$$\mu_3(X) = \begin{cases} 1, & if\ d_p^{PIS^{\alpha-\text{TL}}}(X) < \left(d_p^{PIS^{\alpha-\text{TL}}}\right)^*, \\ 1 - \dfrac{d_p^{PIS^{\alpha-\text{TL}}}(X) - \left(d_p^{PIS^{\alpha-\text{TL}}}\right)^*}{\left(d_p^{PIS^{\alpha-\text{TL}}}\right)^{/} - \left(d_p^{PIS^{\alpha-\text{TL}}}\right)^*}, & if\ \left(d_p^{PIS^{\alpha-\text{TL}}}\right)^- \geq d_p^{PIS^{\alpha-\text{TL}}}(X) \geq \left(d_p^{PIS^{\alpha-\text{TL}}}\right)^*, \\ 0, & if\ d_p^{PIS^{\alpha-\text{TL}}}(X) > \left(d_p^{PIS^{\alpha-\text{TL}}}\right)^-, \end{cases} \quad (20\text{-a})$$

$$\mu_4(X) = \begin{cases} 1, & if\, d_p^{NIS^{\alpha-\text{TL}}}(X) > \left(d_p^{NIS^{\alpha-\text{TL}}}\right)^*, \\ 1 - \dfrac{\left(d_p^{NIS^{\alpha-\text{TL}}}\right)^* - d_p^{NIS^{\alpha-\text{TL}}}(X)}{\left(d_p^{NIS^{\alpha-\text{TL}}}\right)^* - \left(d_p^{NIS^{\alpha-\text{TL}}}\right)^-}, & if \left(d_p^{NIS^{\alpha-\text{TL}}}\right)^- \leq d_p^{NIS^{\alpha-\text{TL}}}(X) \leq \left(d_p^{NIS^{\alpha-\text{TL}}}\right)^*, \\ 0, & if\, d_p^{NIS^{\alpha-\text{TL}}}(X) < \left(d_p^{NIS^{\alpha-\text{TL}}}\right)^-, \end{cases} \quad (20\text{-b})$$

where

$\left(d_p^{PIS^{\alpha-\mathrm{TL}}}\right)^* = \underset{X^/ \in M^{//}}{Minimize}\ d_p^{PIS^{\alpha-\mathrm{TL}}}(X)$ *and the solution is* $X^{PIS^{\alpha-\mathrm{TL}}}$,

$\left(d_p^{NIS^{\alpha-\mathrm{TL}}}\right)^* = \underset{X^/ \in M^{//}}{Maximize}\ d_p^{NIS^{\alpha-\mathrm{TL}}}(X)$ *and the solution is* $X^{NIS^{\alpha-\mathrm{TL}}}$,

$\left(d_p^{PIS^{\alpha-\mathrm{TL}}}\right)^- = d_p^{PIS^{\alpha-\mathrm{TL}}}\left(X^{NIS^{\alpha-\mathrm{TL}}}\right) and \left(d_p^{NIS^{\alpha-\mathrm{TL}}}\right)^- = d_p^{NIS^{\alpha-\mathrm{TL}}}\left(X^{PIS^{\alpha-\mathrm{TL}}}\right)$

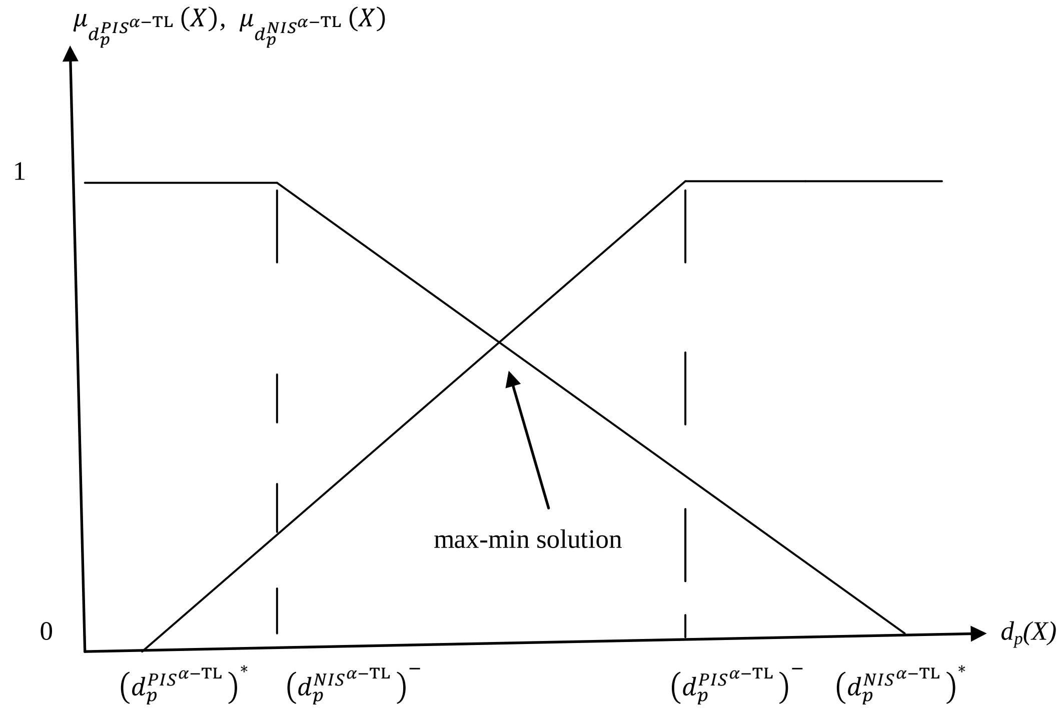

Figure (3): The membership functions of $\mu_{d_p^{PIS^{\alpha-\mathrm{TL}}}}(X)$ *and* $\mu_{d_p^{NIS^{\alpha-\mathrm{TL}}}}(X)$

Now, by applying the max-min decision model which is proposed by R. E. Bellman and L. A. Zadeh [14] and extended by H. –J. Zimmermann [28], we can resolve problem (18). The satisfactory solution of the LS-TL-LMOP Problem, $X^{/*^{\alpha-\mathrm{BL}}}$, may be obtained by solving the following model:

$$\mu_D\left(X^{/*^{\alpha-\mathrm{TL}}}\right) = \underset{X^/ \in M^{//}}{Maximize}\ \{Min.(\mu_3(X),\ \mu_4(X))\} \quad (21)$$

Finally, if $\delta^{\alpha-\mathrm{TL}} = Minimize\ (\mu_3(X),\ \ \mu_4(X))$, the model (21) is equivalent to the form of Tchebycheff model (see [17]), which is equivalent to the following model:

$Maximize\ \delta^{\alpha-\mathrm{TL}}$, (22-a)
subject to
$\mu_3(X) \ge \delta^{\alpha-\mathrm{TL}}$, (22-b)
$\mu_4(X) \ge \delta^{\alpha-\mathrm{TL}}$, (22-c)

$\dfrac{x^/_{I_1 i} - \left(x^{/*^{\alpha-FLDM}}_{I_1 i} - \tau_i^L\right)}{\tau_i^L} \ge \delta^{\alpha-\mathrm{TL}}, i = 1,2,\ldots,n_{I_1} + m_j, j = 1,2,\ldots,q$ (22-d)

$\dfrac{\left(x^{/*^{\alpha-FLDM}}_{I_1 i} + \tau_i^R\right) - x^/_{I_1 i}}{\tau_i^R} \ge \delta^{\alpha-\mathrm{TL}}, i = 1,2,\ldots,n_{I_1} + m_j, j = 1,2,\ldots,q$ (22-e)

$X^/ \in M^{//},\quad \delta^{\alpha-\mathrm{TL}} \in [0,1],$ (22-f)

where $\delta^{\alpha-\mathrm{TL}}$ is the satisfactory level for both criteria of the shortest distance from the PIS and the farthest distance from the NIS. It is well known that if the Pareto optimal solution of (22) is the vector $\left(\delta^{*^{\alpha-\mathrm{TL}}}, X^{/*^{\alpha-\mathrm{TL}}}\right)$, then $X^{/*^{\alpha-\mathrm{TL}}}$ is a nondominated solution of (19) and a satisfactory solution for the LS-TL-LMOP problem.

IV. The Interactive Algorithm of TOPSIS for Solving α-LS-TL-LMOP Problems

Thus, we can introduce the following interactive algorithm to gernerate a set of satisfactory solutions for the α-LS-TL-LMOP problems:

The algorithm (Alg-I):

Phase (0):

Step 0. (i) Elicit a membership function from the decision maker (DM) for each fuzzy number $\tilde{y}$ in problem (1). For example, the fuzzy parameter $\tilde{y}=(y_1,y_2,y_3,y_4)$ can have a membership function of the following form [25, 28]:

$$\mu_{\tilde{y}}(y) = \begin{cases} 0, & y \le y_1, \\ 1-\left[\frac{(y-y_2)}{(y_1-y_2)}\right]^2, & y_1 \le y \le y_2, \\ 1, & y_2 \le y \le y_3, \\ 1-\left[\frac{(y-y_3)}{(y_4-y_3)}\right]^2, & y_3 \le y \le y_4 \\ 0, & y \ge y_4. \end{cases} \quad (23)$$

(ii) Ask the (DM) to select $\alpha = \alpha^* \in [0,1]$.
(iii) Transform problem (1) to the form of problem (4), by using (i) and (ii).
(iv) Transform problem (4) to the form of problem (5).
(v) Transform problem (5) to the form of problem (6).

Phase (I):

Step 1. Construct the PIS payoff table of problem (8) by using the decomposition algorithm [15, 16], and obtain $z^{*^{\alpha-FLDM}} = \left(z_1^{*^{\alpha-FLDM}}, z_2^{*^{\alpha-FLDM}}, \ldots, z_{k_{I_1}}^{*^{\alpha-FLDM}}\right)$ the individual positive ideal solutions.

Step 2. Construct the NIS payoff table of problem (8) by using the decomposition algorithm, and obtain $z^{-^{\alpha-FLDM}} = (z_1^{-^{\alpha-FLDM}}, z_2^{-^{\alpha-FLDM}}, \ldots\ldots, z_{k_{I_1}}^{-^{\alpha-FLDM}})$, the individual negative ideal solutions.

Step 3. Use equations (9 and 10) and the above steps to construct $d_p^{PIS^{\alpha-FLDM}}$ and $d_p^{NIS^{\alpha-FLDM}}$.

Step 4. Transform problem (8) to the form of problem (11).

Step 5. (i) Ask the FLDM to select $p = p^* \in \{1, 2, \ldots, \infty\}$,
(ii) Ask the FLDM to select
$w_i = w_i^*, i = 1,2,\ldots,k_{I_1}$, where $\sum_{i=1}^{k_{I_1}} w_i = 1$,

Step 6. Use step 5, and equation (9) to compute $d_p^{PIS^{\alpha-FLDM}}$ and $d_p^{NIS^{\alpha-FLDM}}$.

Step 7. Construct the payoff table of problem (11):
At $p = 1$, use the decomposition algorithm.
At $p \ge 2$, use the generalized reduced gradient method, [15, 21, 22], and obtain:

$$d_p^{-^{\alpha-FLDM}} = \left(\left(d_p^{PIS^{\alpha-FLDM}}\right)^-, \left(d_p^{NIS^{\alpha-FLDM}}\right)^-\right),$$

$$d_p^{*^{\alpha-FLDM}} = \left(\left(d_p^{PIS^{\alpha-FLDM}}\right)^*, \left(d_p^{NIS^{\alpha-FLDM}}\right)^*\right).$$

Step 8. Construct problem (14) by using the membership functions (12).

Step 9. Solve problem (14) to obtain $\left(\delta^{*^{\alpha-FLDM}}, X^{*^{\alpha-FLDM}}\right)$.

Step 10. Ask the FLDM to select the maximum negative and positive tolerance values τ_i^L and τ_i^R, $i = 1,2,\ldots,n_{I_1}$ on the decision vector $X_{I_1}^{*^{\alpha-FLDM}} = \left(X_{I_1 1}^{*^{\alpha-FLDM}}, X_{I_1 2}^{*^{\alpha-FLDM}}, \ldots, X_{I_1 n_{I_1}}^{*^{\alpha-FLDM}}\right)$.

Phase (II):

Step 11. Construct the PIS payoff table of problem (16) by using the decomposition algorithm [15,16], and obtain $z^{*^{\alpha-SLDM}} = \left(z_1^{*^{\alpha-SLDM}}, z_2^{*^{\alpha-SLDM}}, \ldots, z_{k_{I_2}}^{*^{\alpha-SLDM}}\right)$ the individual positive ideal solutions.

Step 12. Construct the NIS payoff table of problem (16) by using the decomposition algorithm, and obtain $z^{-^{\alpha-SLDM}} = (z_1^{-^{\alpha-SLDM}}, z_2^{-^{\alpha-SLDM}}, \ldots\ldots, z_{k_{I_2}}^{-^{\alpha-SLDM}})$ the individual negative ideal solutions.

Step 13. Use equations (17 and 18) and the above steps (11 &12) to construct $d_p^{PIS^{\alpha-TL}}$ and $d_p^{NIS^{\alpha-TL}}$.

Step 14. Transform problem (6) to the form of problem (19).

Step 15. Ask the FLDM to select $w_i = w_i^*, i = 1,2,\ldots,k$, where $\sum_{i=1}^{k} w_i = 1$,

Step 16. Use steps (5-I, 13 & 15) to compute $d_p^{PIS^{\alpha-TL}}$ and $d_p^{NIS^{\alpha-TL}}$.

Step 17. Construct the payoff table of problem (19):
At p=1, use the decomposition algorithm.
At p≥2, use the generalized reduced gradient method, [21, 22], and obtain:
$d_p^{-^{\alpha-TL}} = \left(\left(d_p^{PIS^{\alpha-TL}}\right)^-, \left(d_p^{NIS^{\alpha-TL}}\right)^-\right)$, $d_p^{*^{\alpha-TL}} = \left(\left(d_p^{PIS^{\alpha-TL}}\right)^*, \left(d_p^{NIS^{\alpha-TL}}\right)^*\right)$.

Step 18. Use equations (15 and 20) to construct problem (28).

Step 19. Solve problem (22) to obtain $\left(\delta^{*^{\alpha-TL}}, X^{*^{\alpha-TL}}\right)$.

Step 20. If the DM is satisfied with the current solution , go to step 21. Otherwise, go to step 0.

Step 21. Stop.

V. An Illustrative Numerical Example

Consider the following LS-TL-LMOP problem with fuzzy parameters in the right-hand side of the independent constraints:

[FLDM]

$$\underset{x_1}{Maximize}\ z_{11}(X) = x_1 - 2x_2 \quad (24-a)$$
$$\underset{x_1}{Minimize}\ z_{12}(X) = 2x_1 + 4x_2 \quad (24-b)$$
$where\ x_1\ solves\ the\ second\ level$

[SLDM]

$$\underset{x_2}{Maximize}\ z_{21}(X) = 2x_1 - 3x_2 \quad (24-c)$$
$$\underset{x_2}{Minimize}\ z_{22}(X) = 3x_1 + 5x_2 \quad (24-d)$$

subject to

$$x_1 + x_2 \le 4, \quad (24-e)$$
$$x_1 + 0.25\, x_2 \le 3, \quad (24-f)$$
$$x_1 - 0.5\, x_2 \ge 1, \quad (24-g)$$
$$x_1 \le 3, \quad (24-h)$$
$$x_1 \ge 1, \quad (24-i)$$
$$x_1 \ge \tilde{y}, \quad (24-j)$$
$$x_2 \le 1.5, \quad (24-k)$$

$x_2 \geq 0.5,$ $(24-l)$

where $\tilde{y}=(0,3,4,5)$.

We shall use (Alg-I) to solve the above problem. Let $\alpha=\alpha^*=0.36$ and by using the membership function (26), thus, we can get the following α-LS-TL-LMOP Problem :

[α-FLDM]

$$\underset{x_1}{Maximize}\, z_{11}(X) = x_1 - 2x_2 \quad (25-a)$$
$$\underset{x_1}{Minimize}\, z_{12}(X) = 2x_1 + 4x_2 \quad (25-b)$$

where x_1 solves the second level

[α-SLDM]

$$\underset{x_2}{Maximize}\, z_{21}(X) = 2x_1 - 3x_2 \quad (25-c)$$
$$\underset{x_2}{Minimize}\, z_{22}(X) = 3x_1 + 5x_2 \quad (25-d)$$

subject to

$x_1 + x_2 \leq 4,$ $(25-e)$

$x_1 + 0.25\, x_2 \leq 3,$ $(25-f)$

$x_1 - 0.5\, x_2 \geq 1,$ $(25-g)$

$x_1 \leq 3,$ $(25-h)$

$x_1 \geq 1,$ $(25-i)$

$x_1 - y \geq 0,$ $(25-j)$

$y \leq 4.6,$ $(25-k)$

$y \geq 2.4,$ $(25-l)$

$x_2 \leq 1.5,$ $(25-m)$

$x_2 \geq 0.5,$ $(25-n)$

Obtain *PIS* and *NIS* payoff tables for the α-*FLDM* of the α-LS-TL-LMOP Problem (25).

Table (1) : PIS payoff table for the α-FLDM of problem (25)

	z_{11}	z_{12}	x_1	x_2	y
$\underset{x_1}{Maximize}\, z_{11}(X)$	1.88^*	7.76	2.88	0.5	2.4
$\underset{x_1}{Minimize}\, z_{12}(X)$	1.4	6.8^*	2.4	0.5	2.4

PIS: $z^{*^{\alpha-\text{FLDM}}}$=(1.88 , 6.8)

Table (2) : NIS payoff table for the α-*FLDM* of problem (25)

	z_{11}	z_{12}	x_1	x_2	y
$\underset{x_1}{Minimize}\, z_{11}(X)$	-0.6^-	10.8	2.4	1.5	2.4
$\underset{x_1}{Maximize}\, z_{12}(X)$	-0.5	11^-	2.5	1.5	2.4

NIS: $z^{-^{\alpha-\text{FLDM}}}$= (-0.6 , 11)

Next, construct equation (10) and obtain the following equations:

$$d_P^{PIS^{\alpha-\text{FLDM}}} = \left[w_1^p \left(\frac{1.88 - z_{11}(X)}{1.88-(-0.6)} \right)^p + w_2^p \left(\frac{z_{12}(x) - (6.8)}{11-(6.8)} \right)^p \right]^{1/p}$$

$$d_P^{NIS^{\alpha-\text{FLDM}}} = \left[w_1^p \left(\frac{z_{11}(X) - (-0.6)}{1.88-(-0.6)} \right)^p + w_2^p \left(\frac{11 - z_{12}(x)}{11-(6.8)} \right)^p \right]^{1/p}$$

Thus, problem (11) is obtained. In order to get numerical solutions, assume that $w_1^p = w_2^p = 0.5$ and $p=2$,

Table (3) : PIS payoff table of problem (11), when $p=2$.

	$d_2^{PIS^{\alpha-\text{FLDM}}}$	$d_2^{NIS^{\alpha-\text{FLDM}}}$	z_{11}	z_{12}	x_1	x_2	y
$Min.\, d_2^{PIS^{\alpha-FLDM}}$	0.0739^*	0.6335^-	1.6004	7.2008	2.6004	0.5	2.6004
$Max.\, d_2^{NIS^{\alpha-FLDM}}$	0.0968^-	0.6423^*	1.4	6.8	2.4	0.5	2.4

$d_2^{*^{\alpha-FLDM}}$=(0.0739 , 0.6423) , $d_2^{-^{\alpha-FLDM}}$=(0.0968 , 0.6335).

Now, it is easy to compute problem (14) :

$Maximize\ \delta^{\alpha-FLDM}$

subject to

$x_1 + x_2 \leq 4,$

$x_1 + 0.25\, x_2 \leq 3,$

$x_1 - 0.5\, x_2 \geq 1,$

$x_1 \leq 3,$

$x_1 \geq 1,$

$x_1 - y \geq 0,$

$y \leq 4.6,$

$y \geq 2.4,$

$x_2 \leq 1.5,$

$x_2 \geq 0.5,$

$$\left(\frac{d_2^{PIS^{\alpha-FLDM}}(X) - 0.0739}{0.0968 - 0.0739} \right) \geq \delta^{\alpha-FLDM},$$

$$\left(\frac{0.6423 - d_2^{NIS^{\alpha-FLDM}}(X)}{0.6423 - b\ 0.6335} \right) \geq \delta^{\alpha-FLDM},$$

$$\delta^{\alpha-\text{FLDM}} \in [0,1] \ .$$

The maximum "satisfactory level" ($\delta^{\alpha-\text{FLDM}}=1$) is achieved for the solution $X_1^{*^{\alpha-FLDM}}=2.4$, $X_2^{*^{\alpha-FLDM}}=1$, $y=2.4$ and $(f_{11}, f_{12}) = (0.4\ ,\ 8.8)$. Let the FLDM decide $X_1^{*^{\alpha-FLDM}}=2.4$ with positive tolerance $\tau^R = 0.5$ (one sided membership function [24,23,25]).

Obtain *PIS* and *NIS* payoff tables for the α-SLDM of the α-LS-TL-LMOP Problem (25).

Table (4) : PIS payoff table for the α-SLDM of problem (25)

	z_{21}	z_{22}	x_1	x_2	y
$\underset{x_2}{Maximize}\, z_{21}(X)$	4.25^*	11.125	2.8750	0.5	2.4
$\underset{x_2}{Minimize}\, z_{22}(X)$	3.3	9.7^*	2.4	0.5	2.4

PIS: $z^{*^{\alpha-SLDM}}=(4.25\ ,\ 9.7)$

Table (5) : NIS payoff table for the α-SLDM of problem (25)

	z_{21}	z_{22}	x_1	x_2	y
$\underset{x_2}{Minimize}\, z_{21}(X)$	0.3^-	14.7	2.4	1.5	2.4
$\underset{x_2}{Maximize}\, z_{22}(X)$	0.5	15^-	2.5	1.5	2.4

NIS: $z^{-^{\alpha-\text{SLDM}}}=(0.3\ ,\ 15)$

Next, compute equation (18) and obtain the following equations:

$$d_P^{PIS^{\alpha-\text{TL}}}=\left[w_1^p\left(\frac{1.88-z_{11}(X)}{1.88-(-0.6)}\right)^p+w_2^p\left(\frac{z_{12}(x)-\ (6.8)}{11-(6.8)}\right)^p+w_3^p\left(\frac{24.25-z_{21}(X)}{4.25-0.3}\right)^p+w_4^p\left(\frac{z_{22}(X)-\ 9.7}{15-9.7}\right)^p\right]^{1/p}$$

$$d_P^{NIS^{\alpha-\text{TL}}}=\left[w_1^p\left(\frac{z_{11}(X)-(-0.6)}{1.88-(-0.6)}\right)^p+w_2^p\left(\frac{11-\ z_{12}(x)}{11-(6.8)}\right)^p+w_3^p\left(\frac{z_{21}(X)-0.3}{4.25-0.3}\right)^p+w_4^p\left(\frac{15-\ z_{22}(X)}{15-9.7}\right)^p\right]^{1/p}$$

Thus, problem (19) is obtained. In order to get numerical solutions, assume that $w_1^p=w_2^p=w_3^p=w_4^p=0.25$ and $p=2$,

Table (3) : *PIS* payoff table of problem (19), when $p=2$.

	$d_2^{PIS^{\alpha-\text{TL}}}$	$d_2^{NIS^{\alpha-\text{TL}}}$	x_1	x_2	y
$Min.\, d_2^{PIS^{\alpha-\text{TL}}}$	0.0581^*	0.4424^-	2.6068	0.5	2.6068
$Max.\, d_2^{NIS^{\alpha-\text{TL}}}$	0.0772^-	0.4491^*	2.4	0.5	2.4

$$d_2^{*^{\alpha-TL}}=(0.0581\ ,\ 0.4491)\ ,\ d_2^{-^{\alpha-TL}}=(0.0772\ ,\ 0.4424).$$

Now, it is easy to compute (22) :

$$Maximize\ \delta^{\alpha-\text{TL}}$$

subject to

$$x_1+x_2\le 5,\ \ x_1\ \le 2,\ \ x_2\le 4,$$
$$x_1,x_2\ge 0\ ,$$
$$\left(\frac{d_2^{PIS^{\alpha-\text{TL}}}(X)-0.0581}{0.0772-0.0581}\right)\ge\delta^{\alpha-\text{TL}}\ ,$$
$$\left(\frac{0.4491-d_2^{NIS^{\alpha-\text{TL}}}(X)}{0.4491-0.4424}\right)\ge\delta^{\alpha-\text{TL}},$$
$$\left(\frac{(2.4+0.5)-x_1}{0.5}\right)\ge\delta^{\alpha-\text{TL}}$$
$$\delta^{\alpha-\text{TL}}\in[0,1]\ \ .$$

The maximum "satisfactory level" ($\delta^{\alpha-\text{TL}}=1$) is achieved for the solution $X_1^{*^{\alpha-\text{TL}}}=2.4$, $X_2^{*^{\alpha-\text{TL}}}=1$.

VI. Concluding Remarks

In this paper, a TOPSIS approach has been extended to solve LS-TL-LMOP problems with fuzzy parameters in the right-hand side of the independent constraints.. The LS-TL-LMOP problems with fuzzy parameters in the right-hand side of the independent constraints using TOPSIS approach provides an effective way to find the compromise (satisfactory) solution of such problems. In order to obtain a compromise (satisfactory) solution to the α −LS-TL-LMOP problems using the proposed TOPSIS approach, a modified formulas for the distance function from the PIS and the distance function from the NIS are proposed and modeled to include all objective functions of both the first and the second levels. Thus, the bi-objective problem is obtained which can be solved by using membership functions of fuzzy set theory to represent the satisfaction level for both criteria and obtain TOPSIS, compromise solution by a second–order compromise. The max-min operator is then considered as a suitable one to resolve the conflict between the new criteria (the shortest distance from the PIS and the longest distance from the NIS). An interactive TOPSIS algorithm for solving these problems are also proposed. An illustrative numerical example is given to demonstrate the proposed TOPSIS approach and the interactive algorithm.

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