An infinity differentiable kernel function and its application in SPH method

Jie Chen, Daming Yuan

***Abstract*— In this paper, we propose a new kernel function W_ε used in the Smoothing Particle Hydrodynamic (SPH) method. Three kernel functions are employed to solve the differential equations. Numerical results show that W_ε has the highest efficency.**

***Index Terms*— kernel function, smoothing particle hydrodynamics method, numerical method**

I. Introduction

The Smoothing Particle Hydrodynamic method is a completely without grid and pure Lagrange particle method gradually developed in the nearly 30 years. It is also a promising numerical calculations[1] and catch widespread concern[2,3] in recent years.

In the SPH method, the value of function $f(\boldsymbol{x})$ at the point *x* can be represent by Integration as

$$f(\boldsymbol{x}) = \int_\Omega f(\boldsymbol{x}')\delta(\boldsymbol{x}-\boldsymbol{x}')d\boldsymbol{x}',$$

(1)

where $\boldsymbol{x} \in R^n, n=1,2,3$ is the space dimension, $\delta(\boldsymbol{x})$ is normally Dirac function.

Replaced $\delta(\boldsymbol{x}-\boldsymbol{x}')$ by the smooth function $W(\boldsymbol{x}-\boldsymbol{x}',h)$, then (1) can be written as

$$f(\boldsymbol{x}) \approx \int_\Omega f(\boldsymbol{x}')W(\boldsymbol{x}-\boldsymbol{x}',h)d\boldsymbol{x}',$$

here $W(\boldsymbol{x}-\boldsymbol{x}',h)$ is also known as kernel or smooth kernel function, *h* is called the smoothing length.

According to the writing habits of SPH, the above kernel approximate operators can be denoted as

$$< f(\boldsymbol{x}) >= \int_\Omega f(\boldsymbol{x}')W(\boldsymbol{x}-\boldsymbol{x}',h)d\boldsymbol{x}'.$$

The derivatives of $f(\boldsymbol{x})$ function is obtained easily by using the divergence theorem. It's worth noting that because of the compactly supported of kernel, all derivatives of $f(\boldsymbol{x})$ function is determined by each order derivative of the kernel function and the value of $f(\boldsymbol{x})$.

In the SPH method, function $f(\boldsymbol{x})$ is discrete as

$$< f(\boldsymbol{x}) >= \sum_{j=1}^{N} \frac{m_j}{\rho_j} f(\boldsymbol{x}_j)\cdot W(x-x_j,h).$$

Jie Chen, The mathematics and information college, Nanchang Hangkong university
Daming Yuan, The mathematics and information college, Nanchang Hangkong university

Where the physical meaning of m_j and $\rho_j\,(j=1,2,\cdots,N)$ is the quality and density of the i-th particle, respectively.

The kernel of SPH method typically required to meet the following conditions:

1. Regularization: $\int_\Omega W(\boldsymbol{x}-\boldsymbol{x}',h)d\boldsymbol{x}'=1$;
2. Limit is Dirac function: $\lim_{h\to 0} W(\boldsymbol{x}-\boldsymbol{x}',h) = \delta(\boldsymbol{x}-\boldsymbol{x}')$;
3. Compact support: $W(\boldsymbol{x}-\boldsymbol{x}',h)=0$ when $|\boldsymbol{x}-\boldsymbol{x}'| > kh$, where *k* is the constant related to the point *x*;
4. The distance $|\boldsymbol{x}-\boldsymbol{x}'|$ between *x* and *x'* is monotonically decreasing and so on.

Many kernel functions satisfied the above conditions have been proposed in the development of SPH method. Such as the commonly used Gaussian kernel $W_G(R,h)$ (abbreviated as W_G), Spline kernel, Index kernel and B-Spline kernel $W_B(R,h)$ (abbreviated as W_B) and so on. We propose a new infinity differentiable kernel function in this paper, and compared with W_G , W_B . To this end, more detailed description of these two kernels are introduced in the following, for other kernel function can refer[1].

The Gaussian kernel $W_G(R,h)$ is defined as[1]

$$W_G = \alpha_d \exp(-R^2).$$

Where the value of α_d is related to the space dimension *n*, i.e.

$$\alpha_d = \begin{cases} \dfrac{1}{\pi^{1/2}h}, & n=1; \\ \dfrac{1}{\pi h^2}, & n=2; \\ \dfrac{1}{\pi^{3/2}h^3}, & n=3. \end{cases}$$

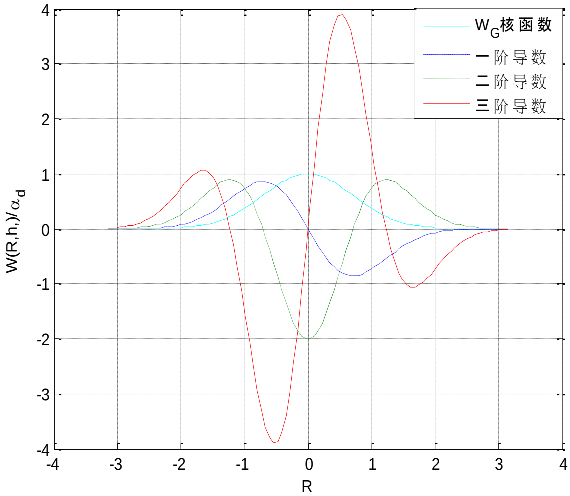
Where $R = \dfrac{|\boldsymbol{x}-\boldsymbol{x}'|}{h}$.

Gaussian kernel W_G is a better choice when it comes to higher derivative because of its Sufficiently smooth, stable and high precision computing. However, due to the support domain radius of Gaussian kernel functions relatively large, therefore, we need a relatively large amount of computation[1] for the numerical solution of partial differential equations.

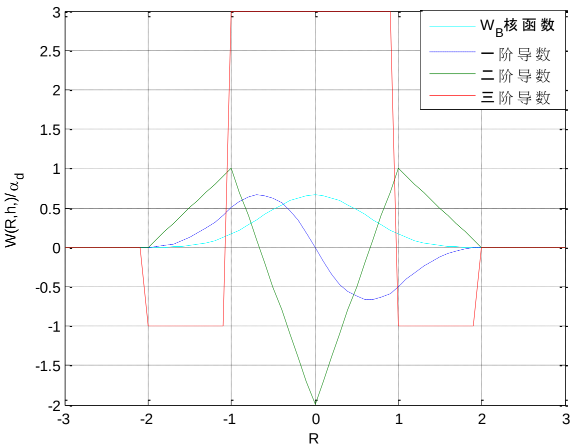
B-Spline kernel $W_B(R,h)$, also known as Cubic spline function, denoted as follows[1]:

$$W_B=\alpha_d\times\begin{cases}\frac{2}{3}-R^2+\frac{R^2}{2}, & 0\le R<1;\\ \frac{(2-R)^3}{6}, & 1\le R<2;\\ 0, & 2\le R.\end{cases}$$

Where the definition of R is the same as W_G, the value of the parameters in one-dimensional, two-dimensional and three-dimensional space is $\frac{1}{h}$, $\frac{15}{7\pi h^2}$, $\frac{3}{2\pi h^3}$, respectively. In figure(1), we give function W_G and W_B and their respective derivative images of the first three in symmetric Interval.



(a) W_G and its derivative



(b) W_B and its derivative

Figure 1

II. A NEW KERNEL FUNCTION

Based on level set methods[4]polished to Heaviside function, i.e.the smoothness of Heaviside function into

$$H_\varepsilon=\frac{1}{\pi}\arctan\frac{x}{h}+\frac{1}{2}.$$

Calculate its derivative $\delta_\varepsilon(x)=\frac{1}{\pi}\frac{\varepsilon^2}{\varepsilon^2+x^2}$. Based on this, a new kernel function is defined in paper[5]

$$W(\boldsymbol{x}-\boldsymbol{x}',\varepsilon)=\begin{cases}\frac{4}{\pi}\cdot\frac{1}{1+R^2}, & 1\le R\le 1;\\ 0, & \{R<0\}\cup\{R>1\}.\end{cases}$$

(2)

Where $R=\frac{|\boldsymbol{x}-\boldsymbol{x}'|}{\varepsilon}$, h is the parameter, its value determines the smoothness of H_ε ,which affecting the size of the support region of $W(\boldsymbol{x}-\boldsymbol{x}',\varepsilon)$. function W_ε and its derivative images of the first three is figure(2). In the following numerical examples we found that the value of h for SPH method is also affected in solving differential equations accuracy. How to get the best value is the content of our further discussion.

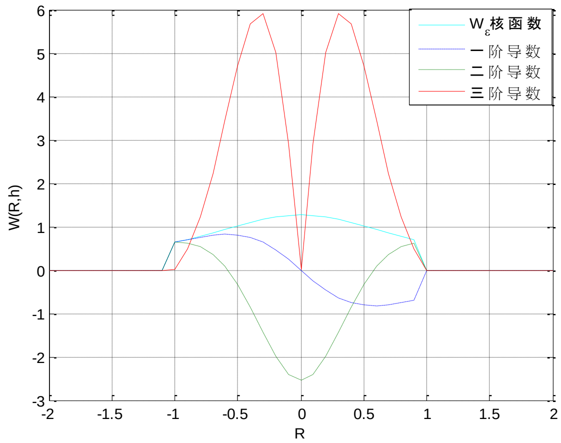


Figure 2 W_ε and its derivative

From the above definition of $W(\boldsymbol{x}-\boldsymbol{x}',\varepsilon)$, it is known that this function maintains relatively good properties of meshless method.

III. THE COMPARISON OF THREE DIFFERENT KERNEL FUNCTIONS FOR SOLVING DIFFERENTIAL EQUATIONS

In order to further study the properties of the kernel function W, We compare SPH method at three kernels, i.e. W_G , W_B , W_ε , for the numerical results when solving differential equations.

Given by boundary value problem of second order ordinary differential equations:

$$\begin{cases}\frac{d^2f(x)}{dx^2}=\sin x, & 0<x<\pi;\\ f(0)=0, & f(\pi)=0.\end{cases}$$

(3)

It's easy to know the analytical solution is $f(x)=-\sin x$. According to the SPH method, equation (3)can be

$$\sum_{j=1}^{n}\frac{m_j}{\rho_j}\cdot f(x_j)\cdot W''(x-x_j,K)=\sin x.$$

Where $x_j(j=1,2,\cdots,n)$ denotes the interval split node, parameters K denotes h or ε .

n	W_ε		W_G	W_B
	ε	error	error	error
250	$0.249h^3$	1.249e-08	0.0100	0.0100
500	$0.124h^3$	8.869e-09	0.0050	0.0050
1000	$0.062h^3$	6.285e-09	0.0025	0.0025
2000	$0.031h^3$	4.452e-09	0.0012	0.0012
4000	$0.016h^3$	3.163e-09	6.192e-04	6.192e-04
8000	$0.008h^3$	2.284e-09	3.096e-04	3.096e-04

Table 1 SPH method at three kernels, i.e. W_G, W_B, W_ε, the compare between error and n of different split interval for the solution of boundary value problem of second order ordinary differential equations.

We first consider the impact of parameter ε on the numerical solution. In figure(3), we compared the analytical solutions and numerical solution of parameters $\varepsilon = 0.249 \times h^3$, $\varepsilon = 0.4 \times h^3$ and $\varepsilon = 0.2 \times h^3$ when $n = 250$. From the graph we can know that numerical solution is very sensitive to parameter ε. How to select the optimal parameters are our future research. Table (1) shows the numerical results for solving boundary value problems when under three kernels. From the table we can see that in the SPH method, by selecting appropriate parameters ε under the same split, the numerical solution of using weight function W_ε has more precision than W_G and W_B.

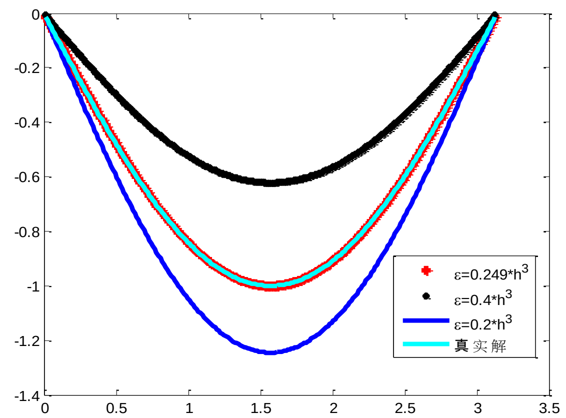


Figure 3 SPH method gets numerical solution when take different parameters in the kernel W_ε

Figure 4 shows point by point error of the SPH method for solving boundary value problem at three kernels.

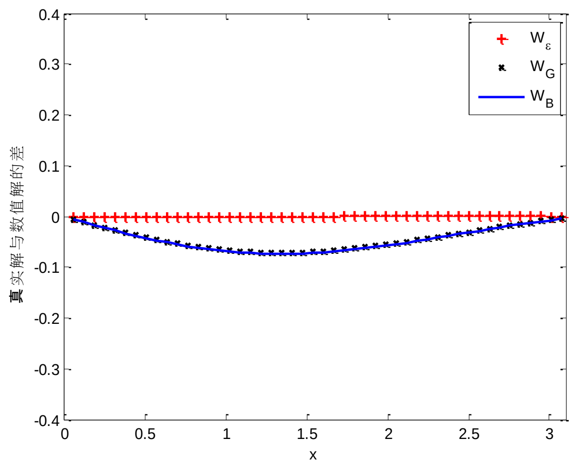


Figure 4 Take the same subdivision interval, point by point error of the SPH method at three kernels

IV. CONCLUSION

This paper compares the SPH method for solving boundary value problems in three different kernel functions. Numerical results show that W_ε can get accurate results when select the appropriate parameter. Kernel W_ε is sensitive to the parameter when applied in SPH method, so how to select the optimal parameter is our work to be carried out next.

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