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On a Fixed Point Theorem in Some None Locally Convex Spaces

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***Abstract*— We extended contraction mapping theorem and proved as a consequences the unique solution of an ordinary linear differential equation in some none convex sets. We also generalized Picard theorem to p-convex sets of ℝ². Further, we extended one of the classical theorems in fixed point theorem**

***Index Terms*—Fixed point theorem, Picard theorem, Existence and Uniqueness theorem, None linear differential equation, p-convex set, None locally convex space.**

I. INTRODUCTION

We generalize the contraction mapping theorem to some none convex sets of ℝ². As a consequence we prove, by using a fixed point theorem, that the differential equation $\frac{dy}{dt} = f(t,y)$ has a unique solution, where f is a continuous mapping from ℝ² to ℝ (theorem (2.1)).

Further, we prove that if $X = \mathbb{R}^2$ is a metric space of a real numbers with a metric defined by a *p*-norm $(1 \geq p > 0)$ and *f* is a differential function of a unit ball *B* of *X*, such that its differential satisfies

$$|Df(z)|^p < 1 \qquad , z \in A_x^y \subset B,$$

then *f* has a fixed point. (theorem (2.2)).

As an application we generalize Picard theorem for a *p*-convex open set. (theorem (2.3)).

II. Existence-Uniqueness Theorem in None Locally Convex Spaces

The following theorem generalizes the contraction mapping theorem to establish an existence-uniqueness theorem for none linear differential equations in some none locally convex cases.

Theorem (2.1) Let f be a continuous mapping from ℝ² to ℝ and satisfies Lipchitz condition with respect to *y*,

$$\|f(t,y) - f(t,z)\| \leq K\|y - z\|^{\frac{1}{p}} \ , 0 < K < 1 \qquad (2.1)$$

in some neighborhood *N* of $(a,b) \in \mathbb{R}^2$, which is p-convex. Then the differential equation

$\frac{dy}{dt} = f(t,y)$, with the initial condition $y(a) = b$
(2.2)

has a unique solution in some neighborhood which may be none convex of a.

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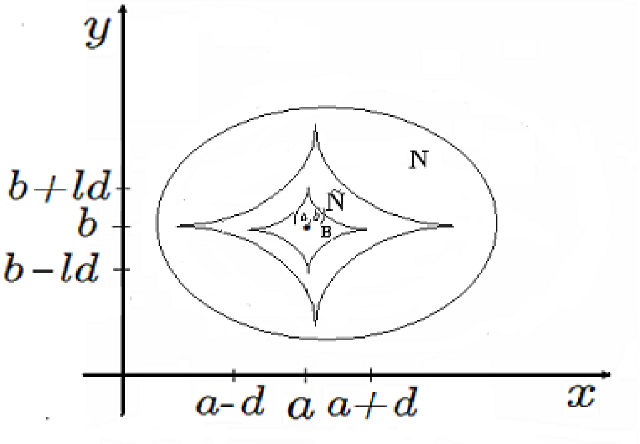

figure (1)

Proof . We observe that (2.2) is equivalent to the integral equation

$$y(t) = b + \int_a^t f(x, y(x))dx. \qquad (2.3)$$

We consider a set M of functions, and a mapping T on M. The image Ty of a function *y* with values $y(x)$ will be given by

$$(Ty)(t) = b + \int_a^t f(x, y(x))dx$$

(2.4)

Let us discuss, how we can find a set of functions which is mapped into itself by *T*. We first choose a compact convex neighborhood Ñ of (a,b) inside *N* and containing *B*, where

$$B = B((a,b), r) = \{(x_1, x_2) \in \mathbb{R}^2 : |x_1 - a|^p + |x_2 - b|^p \leq r\}, \qquad (2.5)$$

$(1 \geq p > 0)$, then *f* is bounded on Ñ, say

$$|f(x,y)| \leq L \quad , (x,y) \in \tilde{N}. \qquad (2.6)$$

If *y* is a function with graph in the *p*-convex neighborhood Ñ (see figure 1) we have

$$|Ty(t) - b| = \left|\int_a^t f(x, y(x))dx\right| \leq \int_a^t |f(x, y(x))|dx$$
$$\leq L\left|\int_a^t dx\right| \leq L\,|t - a|$$

(2.7)

This means that if *y* is a continuous function defined for

$$|t - a| \leq d\,,$$

(2.8)

for which $|y(t) - b| \leq Ld$, then Ty satisfies the same condition. We must choose *d* small enough for the *p*-convex neighborhood (figure 1), and to be in Ñ. We then define *M* to be the set of continuous functions with graphs in *B*, and our argument shows that *M* is mapped into itself by *T*. We use the upper bound norm on *M*. To ensure that *T* is a contraction mapping we should also arrange, in choosing *d*, that we have $dK < 1$, as it will be shown.

Hence we have, for *y* and *z* in *M*,

$$|Ty(t) - Tz(t)| = \left|\int_a^t \left(f(x, y(x)) - f(x, z(x))\right) dx\right|$$

$$\leq \int_a^t |f(x,y(x)) - f(x,z(x))|dx$$
$$\leq d\ sup|f(x,y(x)) - f(x,z(x))|$$

then

$|Ty(\mathrm{t}) - Tz(\mathrm{t})| \leq dK\ sup|y(x) - z(x)|^{1/\mathrm{p}}$,
(2.9)

from (2.1). Consequently

$$\|Ty - Tz\| = sup_{\mathrm{t}}\ |Ty(t) - Tz(t)|$$
$$\leq dK\ sup|y(x) - z(x)|^{1/\mathrm{p}}$$
$$= dK\|y - z\|^{1/\mathrm{p}} \leq dK\ sup|y(x) - z(x)|^{1/\mathrm{p}}$$
$$= dK\|y - z\|^{1/\mathrm{p}}$$

(2.10)

From (2.1), and if $dK < 1$, T is a contraction mapping. Then by the contraction mapping theorem, T has a unique fixed point in M .This means that there is a unique function in M which represents the solution of (2.2). Since any solution of (2.2) is in M (for d sufficiently small), there is a unique solution of (2.2). ■

In what follows we extend one of the classical results in fixed point theorem.

Theorem (2.2) Let $X = \mathbb{R}^2$ be the metric space of real numbers with a metric defined by

$$d(x,y) = \|x - y\|_p = |x_1 - \mathrm{y}_1|^p + |x_2 - \mathrm{y}_2|^p \qquad (0 < p \leq 1), \qquad (2.11)$$

and

$$B = B((0{,}0), 1) = \{(x_1, x_2), |x_1|^p + |x_2|^p \leq 1\}.$$

Consider a differential function

$$f: B \to B$$

such that

$\|Df(z)\|^p \leq K < 1 \qquad , z \in \mathrm{A}_{\mathrm{x}}^{\mathrm{y}} \subset B$
(2.12)

Then f has a fixed point.

Proof. By Lagrange's mean value theorem, given by A. Bayoumi [1], we have for any $x, y \in B$,

$f(x) - f(y) = \sup\|f(z)\|^p\ \|x - y\|$
(2.13)

for some,

$$z \in \mathrm{A}_{\mathrm{x}}^{\mathrm{y}} = \left\{z = \lambda^{\frac{1}{p}}x + (1-\lambda)^{\frac{1}{p}}y\,; 0 \leq \lambda \leq 1\right\} \subseteq B.$$

Hence, by Bayoumi [1, cor.6, p 95]

$$\|f(y) - f(x)\| \leq M\|y - x\|$$
$$d(f(y), f(x)) \leq Md(y,x)$$

(2.14)

If there exists $M > 0$ such that $sup_{z \in \mathrm{A}_{\mathrm{a}}^{\mathrm{b}}}\|Df(z)\| = \|Df(\tilde{x})\|^p \leq M < 1$, for all $\tilde{x} \in A_a^b \subset U$ open in B, then f is a contraction mapping on B. Thus by Banach contraction mapping theorem there exists a unique fixed point $z \in B$, i.e., $f(z) = z$. Therefore, z is the solution of the equation $(z) = z$. ■

We apply now the Banach contraction mapping theorem to prove Picard theorem. Let us first introduce the following definition.

Definition (p-convex set). A set A in a vector space is said to be p-convex $(0 < p \leq 1)$ if for every $, y \in A,\ s,t \geq 0$, we have

$(1-t)^{1/p}x + t^{1/p}y \in \mathrm{A}$ whenever ; $0 \leq t \leq 1$
(2.15)

This is equivalent to saying that, for every $x, y \in A$

$sx + ty \in A$ whenever $\mathrm{s}^{\mathrm{p}} + \mathrm{t}^{\mathrm{p}} = 1$
(2.16)

A is said to be "absolutely p-convex" if for every $x, y \in A$

$$sx + ty \in A$$

for $|s|^p + |t|^p \leq 1$, See Rolewicz [4]. If $p = 1$, we have of course the definitions of convex and balanced convex sets. For example, the unit ball $B_{\ell^p}(0{,}1)$ of space $\ell^{\mathrm{p}}(0 < p \leq 1)$ is p-convex. It is in fact absolutely p-convex, see A. Bayoumi [1].

On the same lines as in theorem (2.1) we prove the following theorem.

Theorem (2.3) (Generalized Picard theorem). Let $f(x,y)$ be a continuous function of two variables in a p-convex set

$$\mathrm{A} = \{(x,y);\ |x|^p + |y|^p \leq 1, 0 < p \leq 1\}$$

and satisfies Lipchitz condition of order 1 in the second variable y. Further, let be (x_o, y_o) an interior of A. Then the differential equation

$$\frac{\mathrm{dy}}{\mathrm{dt}} = f(x,y)$$

(2.17) has a unique solution, say $y = g(x)$ which passes through x_o.

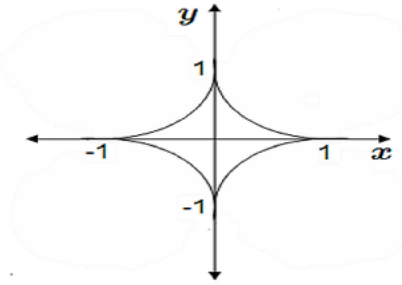

figure (2)

Proof. First of all, we show that the problem of determining the solution of Equation (2.17) is equivalent to the problem of finding the solution of an integral equation. In fact if $y = g(x)$ satisfies equation (2.18) and has the property that $g(x_o) = y_o$, then by integrating equation (2.17) from x_o to x , we get

$$\left.\begin{aligned} g(x) - g(x_o) &= b + \int_{x_o}^{x} f(t, g(t))dt \\ g(x) &= y_o + \int_{x_o}^{x} f(t, g(t))dt \end{aligned}\right\} \qquad (2.18)$$

Thus a unique solution of equation (2.17) is equivalent to a unique solution of equation (2.19). For determining the solution of equation (2.17), we may apply Banach contraction mapping theorem.

Since $f(x,y)$ satisfies the Lipchitz condition of order 1 in y, there exist a constant $q > 0$ such that

$$|f(x,y_1) - f(x,y_2)| \leq q|y_1 - y_2|.$$

(2.19)

Since $f(x,y)$ is a continuous on a compact subset A of $\mathbb{R}^2$, it is bounded and so there exists a positive constant m such that

$$f(x,y) \leq m \qquad \forall x,y \in A$$

(2.20)

Choose a positive constant r such that $rq < 1$ and the rectangle

$$B = \{(x,y)| -r + x_o \leq x < r + x_o, -r_m + y_o \leq y < r_m + y_o\} \qquad (2.21)$$

is contained in A.

Let X be the set of real-valued continuous function $y = g(x)$ defined on $[-r + x_o, r + x_o]$ such that

$$d(g(x), y_o) \leq mr.$$

X is closed subset of the metric space $C_p([x_o - r, x_o + r])$ (denote the set of all real valued continuous functions

$f(x)$ defined on the closed interval $[x_o - r, x_o + r]$ with the distance

$$\rho[f_1(x), f_2(x)] = \rho(f_1, f_2) = sup_{x_o - r \le x \le x_o + r} |f_1(x) - f_2(x)|^p$$

is complete metric space), with sub metric, hence it is a complete metric space since every closed subset of a complete metric space is complete. Let $T: X \to X$ be defined as $Tg = h$, where

$$h(x) = y_o + \int_{x_o}^{x} f(t, g(t))dt$$

Since

$$d(h(x), y_o) = sup_x \left| \int_{x_o}^{x} f(t, g(t))dt \right|^p \le m(x - x_o)^p \le mr \quad ,0 < p \le 1$$

(2.22)

$h(x) \in X$, and so T is well defined. For $g, g_1 \in X$

$$d(Tg, Tg^1) = d(h, h_1) = \sup \left| \int_{x_o}^{x} [f(t, g(t)) - f(t, g_1(t))]dt \right|^p \le \int_{x_o}^{x} sup|f(t, g(t)) - f(t, g_1(t))|^p dt \le q \int_{x_o}^{x} |g(t) - g_1(t)|^p dt \le qr\, d(g, g_1)$$

(2.23)

or

$$d(Tg, Tg_1) \le K\, d(g, g_1)$$

(2.24)

where $0 \le K = qr < 1$. Hence, T is a contraction mapping of X into itself. By Banach contraction mapping theorem T has a unique fixed point $g^* \in X$. This unique fixed point g^*, is the unique solution of equation (2.17). ■

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