Heat Conduction Equation with Newton Type Local and Nonlocal Internal Body Fluxes

Igor Neygebauer, Geminpeter Lyakurwa

***Abstract*— In this work we apply the Newton type of local and nonlocal internal body fluxes to create the constitutive law for the internal body fluxes. We obtain the integral differential equations of the stated heat conduction problem and consider the steady and unsteady linear and nonlinear one dimensional problems. The analytical solutions of the problems are obtained using theoretical analysis, differentiation of the integral differential equation and the method of separation of variables.**

***Index Terms*—Integral differential equation, local and nonlocal internal body fluxes.**

I. INTRODUCTION

The statement of the problems in heat conduction usually includes the surface heat fluxes inside the body and they do not consider the constitutive law for internal body flux [1]-[4].The body fluxes are considered as the internal or external heat sources. Then the linearized theory must accept the nonphysical singularities in temperature field. The introduction of the internal body fluxes allows improving of the heat problems at least in the sense of excluding the nonphysical point singularities.

II. INTERNAL SURFACE AND BODY FLUXES

Consider a body and let us take some control volume, which includes a fixed number of particles. The control volume is surrounded by a control surface. The particles which are inside the control surface are called internal particles and they belong to the control volume. The particles which are outside the control volume are the external particles and they do not belong to the control volume. All other particles belong to the boundary particles of the control volume. There are interactions between particles for example, according to Newton's law of cooling; the resultant of interactions applied to all internal particles of the control volume from the external particles is the internal body flux. The interactions applied to the boundary particles of the control volume from the external particles are the surface fluxes. The Fourier law could be accepted for the internal surface heat fluxes and the Newton's cooling law is taken to describe the nonlocal body heat fluxes.

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III. STATEMENT OF HEAT CONDUCTION PROBLEM

Consider the classical heat conduction problem with the equation according to [1]

$$k\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)-q_0+q_1=c_0\rho\frac{\partial u}{\partial t} \quad (1)$$

where x, y, z are the Cartesian orthogonal coordinates, t is time, $u(x, y, z, t)$ is the temperature, $\rho(x, y, z)$ is the mass-density of the body per unit volume, c_0 is the specific heat, k is the coefficient of thermal conduction, q_0 is a rate of internal body heat flux per unit volume, q_1 is a rate of internal heat generation per unit volume produced in the body.

The introduced in (1) term q_0 could be taken using the Newton's law of cooling in the form of a sum of local and nonlocal fluxes

$$q_0=q_0^{loc}+q_0^{nloc}, \quad (2)$$

where

$$q_0^{loc}=\alpha_1 u(x,y,z), \quad (3)$$

$$q_0^{nloc}=\frac{1}{v^\beta}\int_v \alpha_2\left[u(x,y,z,t)-u(\xi,\eta,\zeta,t)\right]d\xi d\eta d\zeta \quad (4)$$

where v is the volume of the body, α_1,α_2 are constants in Newton's law of cooling, which can depend in general on the coordinates of the body. Let us take them constant, $\beta>0$ is a constant The much more general internal body flux can be taken into account using the similar approach as in [5].

The correspondent initial and boundary conditions should be added to create a well posed initial boundary value problem.

It was shown in [2] that $2D$ and $3D$ heat problems have nonphysical solution in case of a given point boundary condition. This forces to look for a new model which could include the physical solution into consideration.

IV. METHODOLOGY AND RESULTS

In this paper we consider two simple problems in the suggested heat conduction model. The first one is a one dimensional steady state problem and the second one is about a time dependent solution.

Note, that the integral term in differential equations causes difficulties in analysis. Therefore, we have to modify and generalize classical methods. As the result the analytical solutions of the problems are obtained by separation of variables and differentiation of the integral differential equation.

V. Problem 1. Steady State Case

Consider the particular case of the problem stated in section 3. The equation is

$$k\frac{d^2u}{dx^2}-\alpha_1 u-\frac{\alpha_2}{l^\beta}\int_0^l\big(u(x)-u(s)\big)ds=0 \tag{5}$$

where $0\le x\le l$.

The equation (5) can written in the form

$$k\frac{d^2u}{dx^2}-\alpha u+\frac{\alpha_2}{l^\beta}\int_0^l u(s)ds=0, \tag{6}$$

Where

$$\alpha=\alpha_1+\frac{\alpha_2}{l^{\beta-1}}. \tag{7}$$

The boundary conditions are taken as follows

$$u(0)=u_0,\ u(l)=u_1 \tag{8}$$

Differentiating (6) one gets

$$ku''-\alpha u'=0 \tag{9}$$

The general solution of (9) is

$$u=C_2e^{-\sqrt{\frac{\alpha}{k}}x}+C_3e^{\sqrt{\frac{\alpha}{k}}x}+C_1 \tag{10}$$

where C_1, C_2, C_3 are constants of integration.

These constants can be obtained satisfying (6) and the initial conditions (8).

Substituting (10) into (6) one gets

$$-\alpha_1C_1+\frac{\alpha_2}{l^\beta}\int_0^l\left[C_2e^{-\sqrt{\frac{\alpha}{k}}s}+C_3e^{\sqrt{\frac{\alpha}{k}}s}\right]ds=0 \tag{11}$$

If the length is infinite then (11) will be an identity only if

$$C_1=0,C_3=0. \tag{12}$$

Consider the case of finite length. If the length is finite then the integral in (11) is zero and integrating we obtain

$$-mC_1+C_2\left(-e^{-\sqrt{\frac{\alpha}{k}}l}+1\right)+C_3\left(e^{\sqrt{\frac{\alpha}{k}}l}-1\right)=0, \tag{13}$$

where

$$m=\frac{\alpha_1 l^\beta}{\alpha_2}\sqrt{\frac{\alpha}{k}}. \tag{14}$$

Substituting (10) into the boundary conditions (8) we get

$$C_1+C_2+C_3=0, \tag{15}$$

$$C_1+C_2\,e^{-\sqrt{\frac{\alpha}{k}}l}+C_3\,e^{\sqrt{\frac{\alpha}{k}}l}. \tag{16}$$

The system of linear algebraic equations (13), (15), (16) has the following solution

$$C_i=\frac{\Delta_i}{\Delta},i=1,2,3, \tag{17}$$

where

$$\Delta=m\left(e^{-\sqrt{\frac{\alpha}{k}}l}-e^{\sqrt{\frac{\alpha}{k}}l}\right)+2\left(2-e^{-\sqrt{\frac{\alpha}{k}}l}-e^{\sqrt{\frac{\alpha}{k}}l}\right), \tag{18}$$

$$\Delta_1=(u_1+u_0)\left(2-e^{-\sqrt{\frac{\alpha}{k}}l}-e^{\sqrt{\frac{\alpha}{k}}l}\right), \tag{19}$$

$$\Delta_2=\left(e^{\sqrt{\frac{\alpha}{k}}l}-1\right)\big(u_1-u_0-mu_0\big)+m(u_1-u_0), \tag{20}$$

$$\Delta_2=\left(e^{-\sqrt{\frac{\alpha}{k}}l}-1\right)\big(u_1-u_0+mu_0\big)-m(u_1-u_0). \tag{21}$$

The case $l\to\infty$ can be obtained from the solution (10) and the conditions (8) if $C_1=0,C_3=0.$. Then

$$u(x)=C_2e^{-\sqrt{\frac{\alpha_1}{k}}x} \tag{22}$$

Using the conditions (8) we obtain the representation of the solution (10) of the problem (6)-(8) in the form

$$u(x)=u_0e^{-\sqrt{\frac{\alpha_1}{k}}x} \tag{23}$$

and

$$u_1=0. \tag{24}$$

Remark. If $l=\infty$ and $\alpha=0$ then we have the classical case of the problem and the continuous solution in this case does not exist.

VI. Problem 2. The Time Dependent Problem

Consider the equation

$$c_0\frac{\partial u}{\partial t}=k\frac{\partial^2u}{\partial x^2}-\alpha_1u+\frac{\alpha_2}{l^\beta}\int_0^l u(s,t)ds \tag{25}$$

subject to the boundary conditions

$$u(0,t)=u(l,t)=0 \tag{26}$$

and to the initial condition

$$u(x,0)=f(x), \tag{27}$$

where $f(x)$ is the given distribution of the temperature at the time $t=0$.
To obtain the solution of the problem (25)-(27) we apply the method of separation of variables which gives us the representation of the solution in the form

$$u(x,t)=X(x)\cdot T(t) \tag{28}$$

Substituting (28) into (25) we get

$$c_0XT'=kX''T-\alpha XT+\frac{\alpha_2}{l^\beta}\int_0^l XTdx \tag{29}$$

and dividing (29) by XT yields

$$\frac{T'}{T}=\frac{k}{c_0}\frac{X''}{X}-\frac{\alpha}{c_0}+\frac{\alpha_2}{c_0l^\beta X}\int_0^l Xdx=\lambda \tag{30}$$

where λ is a constant.
The boundary conditions (26) imply that

$$X(0)=0,\ X(l)=0 \tag{31}$$

To determine λ consider (30).

$$kX''-\alpha X+\frac{\alpha_2}{l^\beta}\int_0^l Xdx=c_0\lambda X \tag{32}$$

Differentiating (32) we obtain

$$kX'''-(\alpha+c_0\lambda)X'=0 \tag{33}$$

Now we consider (33) subject to the conditions (31). The following cases are possible:
a) Let $\alpha+c_0\lambda<0$.
Denote $k_1=\sqrt{-\frac{\alpha+c_0\lambda}{k}}$, then the solution of (33) is

$$X=C_1+C_2\cos k_1x+C_3\sin k_1x \tag{34}$$

Substituting the solution (34) into (32) we get

$$-(\alpha_1+c_0\lambda)C_1+\frac{\alpha_2}{l^\beta}\int_0^l\left[C_2\cos k_1x+C_3\sin k_1x\right]dx=0 \tag{35}$$

or

$$-mC_1+C_2\sin k_1l-C_3(\cos k_1l-1)=0, \tag{36}$$

where

$$m=\frac{(\alpha_1+c_0\lambda)l^\beta k_1}{\alpha_2}. \tag{37}$$

The conditions (31) imply that

$$C_1+C_2=0, \tag{38}$$

$$C_2\cos k_1l+C_3\sin k_1l+C_1=0. \tag{39}$$

Consider the determinant of the system of linear homogeneous equations (36), (38), (39).

$$\begin{vmatrix} -m & \sin k_1l & -\cos k_1l+1 \\ 1 & 1 & 0 \\ 1 & \cos k_1l & \sin k_1l \end{vmatrix}=0. \tag{40}$$

The equation (40) yields

$$-\sin k_1l\left(m+2\tan\frac{k_1l}{2}\right)=0 \tag{41}$$

and the roots of (41) can be found analytically

$$\sin k_1l=0,\ k_1l=\pi n_1, n_1=0,1,2,...$$
$$\lambda_{n_1}=-\frac{\pi^2kn_1^2+\alpha l^2}{c_0l^2}, n_1=0,1,2,... \tag{42}$$

and numerically

$$\tan\frac{k_1l}{2}=-\frac{m}{2}, \tag{43}$$

$$\lambda_{n_2}=-\frac{kk_1^2+\alpha}{c_0}, n_2=0,1,2,... \tag{44}$$

Then we obtain $C_2=-C_1$ and C_3 is an arbitrary constant.

$$C_1=C_3\frac{\sin k_1l}{\cos k_1l-1}. \tag{45}$$

Then the solution of the problem (31), (32) consists of two sets of functions

$$X_{n_1}(x)=C_{n_1}\sin\frac{\pi n_1x}{l}, n_1=0,1,2,... \tag{46}$$

and

$$X_{n_2}=C_{n_2}\left[\frac{m}{2}\left(1-\cos N_{n_2}x\right)+\sin N_{n_2}x\right], \tag{47}$$

where

$$N_{n_2}=\sqrt{-\frac{\alpha+c_0\lambda_{n_2}}{k}}, n_2=0,1,2,... \tag{48}$$

b) Let $\alpha+c_0\lambda=0$.
Then (33) becomes $X'''=0$ and

$$X(x)=C_1+C_2x+C_3x^2 \tag{49}$$

Substituting (49) into (32) we get

$$\frac{2kl^{\beta}}{\alpha_1}C_3+\frac{\alpha_2}{l^{\beta}}\int_0^l\left(C_1+C_2x+C_3x^2\right)dx=0 \quad (50)$$

or

$$C_1+C_2\frac{l}{2}+C_3\left(\frac{l^2}{3}+\frac{2kl^{\beta-1}}{\alpha_2}\right)=0. \quad (51)$$

The boundary conditions (31) imply

$$C_1=0, \quad (52)$$

$$C_1+C_2l+C_3l^2=0. \quad (53)$$

The solution of the linear algebraic system of equations (51), (52), (53) is

$$C_1=0, C_2=0, C_3=0 \quad (54)$$

and only zero solution exists in this case

$$X(x)=0. \quad (55)$$

c) Let $\alpha+c_0\lambda>0$.

Denote $k_2=\sqrt{\dfrac{\alpha+c_0\lambda}{k}}$, then the general solution of (33) is

$$X=C_1+C_2e^{k_2x}+C_3e^{-k_2x} \quad (56)$$

Substituting (56) into (32) we get

$$-\frac{(\alpha_1+c_0\lambda)l^{\beta}}{\alpha_2}C_1+C_2\int_0^l e^{k_2x}dx+C_3\int_0^l e^{-k_2x}dx=0 \quad (57)$$

or

$$-\frac{(\alpha_1+c_0\lambda)k_2l^{\beta}}{\alpha_2}C_1+C_2(e^{kl}-1)-C_3(e^{-kl}-1)=0. \quad (58)$$

The boundary conditions (31) are

$$C_1+C_2+C_3=0, \quad (59)$$

$$C_1+C_2e^{k_2l}+C_3e^{-k_2l}=0. \quad (60)$$

The dete3rminant of the system of linear algebraic equations (58), (59), (60) equals

$$D=(1-e^{k_2l})\left[2(e^{k_2l}-1)+\frac{(\alpha_1+c_0\lambda)k_2l^{\beta}}{\alpha_2}(e^{k_2l}+1)\right]>0.$$

The solution of the linear algebraic system of equations (58), (59), (60) is

$$C_1=0, C_2=0, C_3=0 \quad (61)$$

and only zero solution exists in this case

$$X(x)=0. \quad (62)$$

Now find the correspondent functions $T_n(t)$ using (30) and

$$\lambda_{n_1},\lambda_{n_2} \quad (63)$$

according to (42), (44).

We have $T_n'-\lambda_nT_n=0$ and

$$T_n(t)=A_ne^{\lambda_nt}. \quad (64)$$

Then the following formal solution of the problem (25) -(26) is obtained

$$u(x,t)=\sum_{n_1=0}^{\infty}C_{n_1}\left(\sin\frac{\pi n_1x}{l}\right)e^{\lambda_{n_1}t}+$$

$$+\sum_{n_2=0}^{\infty}C_{n_2}\left[\frac{m}{2}\left(1-\cos N_{n_2}x\right)+\sin N_{n_1}x\right]e^{\lambda_{n_2}t}. \quad (65)$$

The initial condition (27) should be used to obtain the constants C_{n_1},C_{n_2}. Then (27) and (65) imply that

$$f(x)=\sum_{n_1=0}^{\infty}C_{n_1}\left(\sin\frac{\pi n_1}{l}x\right)+$$

$$+\sum_{n_2=0}^{\infty}C_{n_2}\left[\frac{m}{2}\left(1-\cos N_{n_2}x\right)+\sin N_{n_1}x\right]. \quad (66)$$

The Fourier coefficients are as follows

$$C_{n_1}=\frac{\int_0^l f(x)\sin\frac{\pi n_1x}{l}dx}{\int_0^l\sin^2\frac{\pi n_1x}{l}dx}, \quad (67)$$

$$C_{n_2}=\frac{\int_0^l f(x)\left[\frac{m}{2}\left(1-\cos N_{n_2}x\right)+\sin N_{n_2}x\right]dx}{\int_0^l\left[\frac{m}{2}\left(1-\cos N_{n_2}x\right)+\sin N_{n_2}x\right]^2dx}.$$

VII. Conclusion

The main results of this work are the obtained analytical explicit forms (10), (65) of the solutions of the new statements of the heat conduction problem.

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