New approach for calculating hydraulic radius for partially filled circular conduit by using integration

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***Abstract*— This paper is accomplished to develop new mathematical approach to calculate hydraulic radius(R) for partially filled circular pipe(sewer) with water or sewage by integration .An example is solved by using this method .As comparison with previous method, the results were accurate and this approach is suitable for all variable depths in a pipe .**

***Index Terms*— calculating hydraulic radius, sewers, wetted perimeter**

I. Introduction

Most sewers are designed to flow as open channel and not under pressure. Even though may flow fall at times(1).The condition of partially filled conduit are frequently encountered in environmental engineering practicalary in the case of sewer lines. In a conduit flowing partly full, the fluid is at atm pressure and flow is the same as in open channel (2).It is frequently necessary to determine the velocity and depth of sewage in a pipe which is flowing partially full(2).Several equations are used to this purpose which are Chezy equation, Hazen-williams equation and Manning equation. Manning's equation is used widely to determine velocity through the pipes as shown below:

$$V = \frac{1.486}{n} R^{\frac{2}{3}} S^{\frac{1}{2}}$$

,where :
V:Mean velocity (ft/s).
n: Manning's coefficient of roughness.
R:Hydraulic radius ,the hydraulic radius is defined as the water (or sewage) cross-sectional area (A) divided by wetted perimeter (P)).For circular pressure pipe ,if (D) is the diameter of the pipe the radius (R) is :

$$R = \frac{A}{P} = \frac{\frac{\pi}{4}D^2}{\pi D} = \frac{D}{4} \text{ (ft)}$$

The hydraulic radius (D/4) is correct for two cases:1)when the pipe is full flow ,and 2)when the pipe is half flow.
In the case of the sewage is upper or lower than the diameter horizontal level ,the hydraulic radius (R) will not equal to (D/4) and then mathematical solution for this problem should be required to calculate the flow area (A) and wetted perimeter (P) and then hydraulic radius (R).
S:Slope of energy gradient line (ft/ft).

II. Previous Mathematical Solution

A mathematical solution for calculating hydraulic radius (R) for flow in partially filled circular pipe under horizontal level is presented by Shun Dar Lin (2) as illustrated below :A -schematic pipe cross section is shown in figure(1).The angle(θ) ,flow area(A) ,wetted perimeter (P) and hydraulic radius (R) can be determine by the following equation:

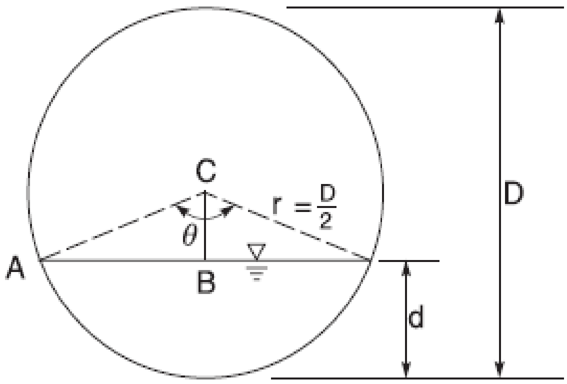

Figure (1) flow in partially filled circular pipe(2)

For angle

$$\cos\frac{\theta}{2} = \frac{\overline{BC}}{\overline{AC}} = \frac{r-d}{r} = 1 - \frac{d}{r} = 1 - \frac{2d}{D}$$

$$\theta = 2\cos^{-1}\left(1 - \frac{2d}{D}\right)$$

Area of triangle ABC, a

$$a = \frac{1}{2}\overline{AB}\,\overline{BC} = \frac{1}{2} r \sin\frac{\theta}{2} r \cos\frac{\theta}{2} = \frac{1}{2} r^2 \frac{\sin\theta}{2}$$
$$= \frac{1}{2}\frac{D^2}{4}\frac{\sin\theta}{2}$$

Flow area A

$$A = \frac{\pi D^2}{4}\frac{\theta}{360} - 2a$$
$$= \frac{\pi D^2}{4}\frac{\theta}{360} - 2\left(\frac{1}{2}\frac{D^2}{4}\frac{\sin\theta}{2}\right)$$
$$A = \frac{D^2}{4}\left(\frac{\pi\theta}{360} - \frac{\sin\theta}{2}\right)$$

For wetted perimeter P

$$P = \pi D \frac{\theta}{360}$$

For hydraulic radius R

$$R = A/P$$

Thus , the flow area (A) , the wetted perimeter (P) and then hydraulic radius (R) can mathematically calculated (2).

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III. NEW APPROACH FOR CALCULTING HYDRAULIC RADIUS (R) BY INTEGRATION

The new approach for calculating hydraulic radius (R)by integration gives exact results compared with the equation presented by Shun Dar Lin. A -schematic pipe cross section is shown in figure(2) to drive a new approach .The flow area(A) ,wetted perimeter (P) and hydraulic radius (R) can be determine by the following equations:

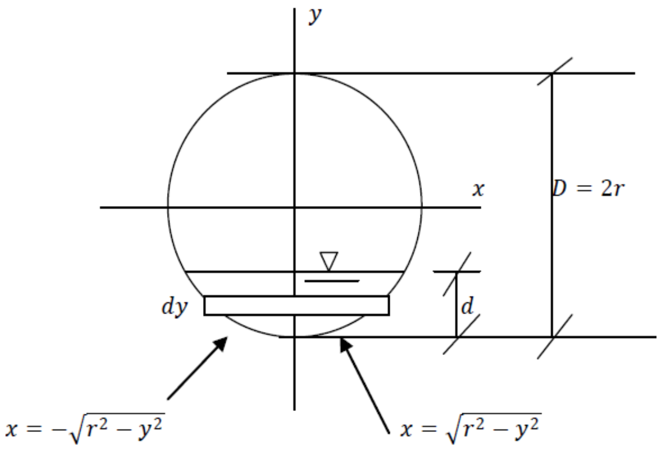


Figure (2) flow in partially filled circular pipe to drive a new approach

The hydraulic radius $R = \frac{A}{P}$

First, a formula of the flow area (A) is derived :

$$A = \int_a^b f(y)dy$$

The equation of the circle with radius (r)and its center at the origin is $x^2 + y^2 = r^2$

$$x = \mp\sqrt{r^2 - y^2} \quad , \quad f(y) = x = \mp\sqrt{r^2 - y^2}$$

$$A = \int_a^b f(y)\,dy = \int_{y_1=-r}^{y_2=-(r-d)} [(\sqrt{r^2 - y^2}) - (-\sqrt{r^2 - y^2})]dy$$

$A = 2\int_{y_1=-r}^{y_2=-(r-d)} \sqrt{r^2 - y^2}\,dy$,using methods of integration by subsitute $y = r\sin\theta$

$$y = r\sin\theta \quad , \quad \sin\theta = \frac{y}{r} \quad , \quad \theta = \sin^{-1}\frac{y}{r}$$

$$\text{when } y_1 = -r \quad , \quad \theta_1 = \sin^{-1}(-1) = \frac{-\pi}{2}$$

$$\text{when } y_2 = -(r-d)\ , \ \theta_2 = \sin^{-1}\frac{y}{r} = \sin^{-1}\frac{-(r-d)}{r}$$

$$dy = r\cos\theta\, d\theta$$

$$A = 2\int_{y_1=-r}^{y_2=-(r-d)} \sqrt{r^2 - y^2}\,dy$$

$$= 2\int_{\theta_1=\frac{-\pi}{2}}^{\theta_2=\sin^{-1}\frac{d-r}{r}} \sqrt{r^2 - (r\sin\theta)^2}\, r\cos\theta\, d\theta$$

$$A = 2r^2\int_{\theta_1=\frac{-\pi}{2}}^{\theta_2=\sin^{-1}\frac{d-r}{r}} \cos^2\theta d\theta$$

$$A = 2r^2\int_{\theta_1=\frac{-\pi}{2}}^{\theta_2=\sin^{-1}\frac{d-r}{r}} \frac{1}{2}(1 + \cos 2\theta)\, d\theta$$

$$A = r^2\left(\theta + \frac{\sin 2\theta}{2}\right), [\theta_1 = \frac{-\pi}{2}\ , \theta_2 = \sin^{-1}\frac{d-r}{r}]$$

$$A = r^2(\theta + \sin\theta\cos\theta)\ \ , [\theta_1 = \frac{-\pi}{2}\ , \theta_2 = \sin^{-1}\frac{d-r}{r}]$$

$$A = r^2\left[\sin^{-1}\frac{y}{r} + \frac{y}{r^2}\sqrt{r^2 - y^2}\right]\ , [y_1 = -r\ , y_2 = -(r-d)]$$

$$A = r^2\left[\left(\sin^{-1}\frac{d-r}{r} + \frac{(d-r)}{r^2}\sqrt{r^2 - (d-r)^2}\right) - \left(\sin^{-1}\frac{-r}{r} + \frac{(0)}{r^2}\right)\right]$$

$$A = r^2\left[\left(\sin^{-1}\frac{d-r}{r} + \frac{(d-r)}{r^2}\sqrt{r^2 - (d-r)^2}\right) - \left(\sin^{-1}\frac{-r}{r} + 0\right)\right]$$

$$A = r^2\left[\left(\sin^{-1}\frac{d-r}{r} + \frac{(d-r)}{r^2}\sqrt{r^2 - (d-r)^2}\right) - \left(\frac{-\pi}{2}\right)\right]$$

$$A = r^2\left[\left(\sin^{-1}\frac{d-r}{r} + \frac{(d-r)}{r^2}\sqrt{r^2 - (d-r)^2}\right) + \left(\frac{\pi}{2}\right)\right]$$

The area of flow at any depth can be calculated by driving formula

$$A = r^2\left[\left(\sin^{-1}\frac{d-r}{r} + \frac{(d-r)}{r^2}\sqrt{r^2 - (d-r)^2}\right) + \left(\frac{\pi}{2}\right)\right]$$

Second, the formula of the wetted perimeter (P) is derived :

$$P = L = \int_a^b \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\,dy$$

The equation of the circle with radius (r)and its center at the origin is $x^2 + y^2 = r^2$

$$x = \mp\sqrt{r^2 - y^2} \quad , \quad f(y) = x = \mp\sqrt{r^2 - y^2}$$

the length of one part can be fined and then multiplying the result by (2)

take $f(y) = x = +\sqrt{r^2 - y^2}$, $\frac{dx}{dy} = \frac{-y}{\sqrt{r^2 - y^2}}$

$$P = L = \int_a^b \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\,dy = \int_{y_1=-r}^{y_2=(r-d)} \sqrt{1 + \left(\frac{-y}{\sqrt{r^2 - y^2}}\right)^2}\,dy$$

$$P = L = \int_{y_1=-r}^{y_2=(r-d)} \frac{r}{\sqrt{r^2 - y^2}}\,dy$$

using methods of integration by subsitute $y = r\sin\theta$

$$y = r\sin\theta \quad , \quad \sin\theta = \frac{y}{r} \quad , \quad \theta = \sin^{-1}\frac{y}{r}$$

$$\text{when } y_1 = -r \quad , \quad \theta_1 = \sin^{-1}(-1) = \frac{-\pi}{2}$$

$$\text{when } y_2 = -(r-d)\ , \ \theta_2 = \sin^{-1}\frac{y}{r} = \sin^{-1}\frac{-(r-d)}{r}$$

$$dy = r\cos\theta\, d\theta$$

$$P = L = \int_{y_1=-r}^{y_2=-(r-d)} \frac{r}{\sqrt{r^2 - y^2}}\,dy$$

$$= \int_{\theta_1=\frac{-\pi}{2}}^{\theta_2=\sin^{-1}\frac{d-r}{r}} \frac{r}{\sqrt{r^2 - (r\sin\theta)^2}}\, r\cos\theta\, d\theta$$

$$P = L = \int_{\theta_1=\frac{-\pi}{2}}^{\theta_2=\sin^{-1}\frac{d-r}{r}} r d\theta$$

$$P = L = r\left[\left(\sin^{-1}\frac{d-r}{r}\right) - \left(\frac{-\pi}{2}\right)\right]$$

$P = L = r\left[\left(\sin^{-1}\frac{d-r}{r}\right) + \left(\frac{\pi}{2}\right)\right]$ wetted perimeter (P) for one part

Total wetted perimeter (P) is multiplying by (2)

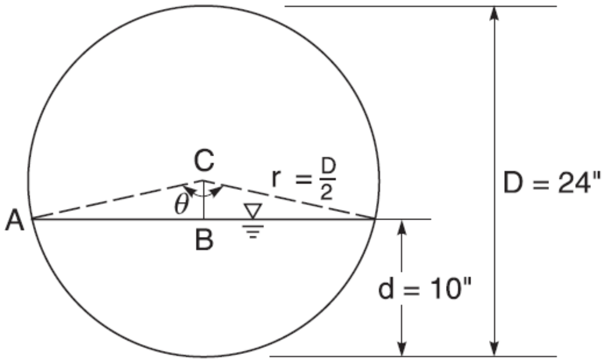
$$P = L = 2r\left[\left(\sin^{-1}\frac{d-r}{r}\right) + \left(\frac{\pi}{2}\right)\right]$$

Now the hydraulic radius at any depth can be calculated :

$$R = \frac{A}{P} = \frac{r^2\left[\left(\sin^{-1}\frac{d-r}{r} + \frac{(d-r)}{r^2}\sqrt{r^2-(d-r)^2}\right)+\left(\frac{\pi}{2}\right)\right]}{2r\left[\left(\sin^{-1}\frac{d-r}{r}\right)+\left(\frac{\pi}{2}\right)\right]}$$

IV. NUMERICAL EXAMPLE

Example: Assume a 24-in diameter sewer concrete pipe (n=0.012) is placed on a slope 0f 2.5in 1000.The depth of the sewer flow is 10-in.What is the average velocity?



1-Solution by Shun Dar Lin method:

$$\cos\frac{\theta}{2} = 1 - \frac{2d}{D} = 1 - \frac{2 \times 10}{24}$$

$$= 0.1667$$

$$\frac{\theta}{2} = \cos^{-1}(0.1667) = 80.4^\circ$$

$$\theta = 160.8^\circ$$

$$D = 24\text{in} = 2 \text{ ft}$$

$$A = \frac{D^2}{4}\left(\pi\frac{\theta}{360} - \frac{\sin\theta}{2}\right)$$

$$= \frac{2^2}{4}\left(3.14 \times \frac{160.8}{360} - \frac{1}{2}\sin 160.8^\circ\right)$$

$$= 1.40 - 0.16$$

$$= 1.24 \ (\text{ft}^2)$$

$$P = \pi D\frac{\theta}{360} = 3.14 \times 2 \text{ ft} \times \frac{160.8}{360}$$

$$= 2.805 \text{ ft}$$

Calculate R

$$R = A/P = 1.24 \text{ ft}^2/2.805 \text{ ft}$$

$$= 0.442 \text{ ft}$$

Calculate V

$$V = \frac{1.486}{n}R^{2/3}S^{1/2}$$

$$= \frac{1.486}{0.012}(0.442)^{2/3}(2.5/1000)^{1/2}$$

$$= 123.83 \times 0.58 \times 0.05$$

$$= 3.591 \ (\text{ft/s})$$

2-Solution by the new approach:

$$A = r^2\left[\left(\sin^{-1}\frac{d-r}{r} + \frac{(d-r)}{r^2}\sqrt{r^2-(d-r)^2}\right)+\left(\frac{\pi}{2}\right)\right]$$

$$A = 12^2\left[\left(\sin^{-1}\frac{10-12}{12} + \frac{(10-12)}{12^2}\sqrt{12^2-(10-12)^2}\right)+\left(\frac{\pi}{2}\right)\right]$$

$$A = 178.418 in^2$$

$$A = \frac{178.418}{12^2} \cong 1.24 ft^2$$

$$P = L = 2r\left[\left(\sin^{-1}\frac{d-r}{r}\right)+\left(\frac{\pi}{2}\right)\right]$$

$$P = 2 \times 12\left[\left(\sin^{-1}\frac{10-12}{12}\right)+\left(\frac{\pi}{2}\right)\right]$$

$$P = 33.68 in$$

$$P = \frac{33.68}{12} \cong 2.805 \ ft$$

$$R = \frac{A}{P} = \frac{1.24}{2.805} = 0.442 ft$$

$$V = \frac{1.486}{n}R^{\frac{2}{3}} \ S^{\frac{1}{2}}$$

$$V = \frac{1.486}{0.012}(0.442)^{\frac{2}{3}}\left(\frac{2.5}{1000}\right)^{\frac{1}{2}} = 3.591\frac{ft}{s}$$

V. CONCLUSION

1)The new mathematical approach to calculate hydraulic radius is very accurate and it may used as alternative in lieu of previous mathematical equations.

2)The new mathematical approach can be used for different depths of sewages in pipe.

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