An Easy New Approach for Matrix Decomposition

Mohammed Hassan Elzubair

***Abstract*— In the mathematical discipline of linear algebra there are many operations defined on matrices such as addition , multiplication and matrix inverse etc. Actually we note that in the real world it is not feasible for most of the matrix computations to be calculated in an optimal explicit way, such as matrix inversion . Matrix decomposition is a fundamental theme in linear algebra and applied statistics which has both scientific and engineering significance. Now suppose we are given a matrix A and we have to transform it into a product of two matrices ,say we want to write A=BC therefore in this case a numerically stable and fairly fast scheme is desired to compute the two matrices A and B. In general we mean by the sentence matrix decomposition or matrix factorization the factorization of a matrix into a product of two or more matrices, so the process of converting or transforming a a given matrix in one side to product of two or more matrices in the other side is called matrix decomposition or matrix factorization. Really matrix factorization or matrix decomposition has numerous algorithm like LU factorization ,QR decomposition ,rank factorization, singular value factorization etc . But most of these algorithms are complicated ,so our goal here is to find a simple algorithm with which we can factorize or decompose the specified matrix and this was done by making use of the inverse of the matrix . Sometimes we need to factorize a matrix according to different constraints like non- negative matrix factorization and this happened if we have a non-negative matrix and we want to factorize it also into non- negative ones, or any other constraints due to the nature of the problem which was represented by the matrices ,some methods of decomposition factorizes approximately not exactly but here in our easy new approach the factorization will be exact .**

In this paper we will introduce a very simple algorithm with which we can factorize any matrix exactly to a product of any numbers of matrices by factorizing again each matrix of matrices which form the product .

***Index Terms*— Invertible matrix, matrix decomposition, noisy matrix, the entries of the matrix, singular matrix, constraints ,matrix multiplication ,associative law, the rank of the matrix.**

I. INTRODUCTION

This paper propose an easy new method for matrix decomposition and it contains four sections, in section 2 there is a theorem with its proof using the notion of the determinant and of the matrix and its properties, in section 3the invertible matrix play an important role for matrix decomposition besides the associative law of matrix multiplication, it also contains some numerical example to make the idea very clear, section 4 is the conclusion of the paper.

We note matrix factorization refers to transformation of a given matrix into a given multiplication form [1].
Now let matrix A be the product of the matrices W and H,

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Mohammed Hassan Elzubair , Taif University - Raniah Branch , Department of Mathematics

A=WH

Matrix multiplication can be implemented as computing the columns vectors of A as linear combinations of the columns vector in W using coefficients supplied by the columns of H, so that each column of A can be computed as follows

$$a_i = w_i h_i \ ,$$

Where a_i is the ith column vector of the matrix A and h_i is the ith column vector of the matrix H .
To make it clear we will give a common example which is facial image decomposition[2].
Let A a matrix with a real-valued, nonnegative entries of size $m \times n$ such that

$$A = WH$$

Where W is $m \times r$, H is $r \times n$, and both factors have none negative entries . If the columns of A are pixels of a facial image, the columns of W may be facial features such as eyes or ears , and the coefficients in H represents the intensity of these features . Non-negative matrix factorization is used in abroad range of applications including graph clustering [3], Protein sequence motif discovery [4], and hyper spectra unmixing [5] .
For factor analyses which is a model for real- valued data we can construct an equation of four matrices in the form $A = WH + E$ where is E called a noisy matrix or the error matrix ,and we can use the least square method to find the minimum error so as to get the approximate equation [7].

$$A \approx WH$$

II. PRELIMINARIES

Sometimes matrix decomposition has a benefit that it makes to reduce dataset and summarization .For example if we have an 1000×50 matrix A this matrix has 50000 entities ,if we success to factorize the matrix A into a product of two matrices W and H such that W is an 1000×2 matrix and H is 2×50 matrix , we note that the sum of the entities in the two matrices is 2100 ,

Theorem (1)
Any square $n \times n$ matrix A which was written as $A = BC$ such that , B is an $n \times k$ matrix and C is an $k \times n$ matrix , $k < n$ then A is singular matrix

Proof

Let ,

$$B = \begin{bmatrix} b_{11} & . & . & . & b_{1k} \\ . & . & . & . & . \\ . & . & . & .. & . \\ . & . & . & . & . \\ b_{n1} & . & . & . & b_{nk} \end{bmatrix}$$

$$C = \begin{bmatrix} c_{11} & . & . & . & c_{1n} \\ . & . & . & . & . \\ . & . & . & .. & . \\ . & . & . & . & . \\ c_{k1} & . & . & . & c_{kn} \end{bmatrix}$$

Since $k < n$ we can add $n-k$ zero columns to the matrix B to get a new square $n \times n$ matrix W, and we can also add $n-k$ zero rows or any $n-k$ rows to get another new square $n \times n$ matrix H

Now we have

$$W = \begin{bmatrix} b_{11} & . & . & . & b_{1k} & 0...0 \\ . & . & . & . & . & \\ . & . & . & .. & . & .. \\ . & . & . & . & . & \\ b_{n1} & . & . & . & b_{nk} & 0...0 \end{bmatrix} ,$$

$$H = \begin{bmatrix} c_{11} & \cdots & & . & c_{1n} \\ . & . & . & . & . \\ .c_{k1} & . & . & .. & .c_{kn} \\ 0 & . & . & . & .0 \\ 0 & . & . & . & o \end{bmatrix}$$

Now also we have $A = WH$.

Also we have $\det(W) = 0$, $\det(H) = 0$.

So we get $\det(A) = \det(W) \times \det(H) = 0 \times 0 = 0 \Rightarrow$ A is singular.

III. METHODOLOGY

In this paper we want to factorize a matrix into two matrix or more using the inverse matrix, so we make use the above theorem

Now suppose we have an $m \times n$ matrix A and we want to decompose it into a product of a certain $m \times m$ invertible matrix D,then we can write

$A = DD^{-1}A \Rightarrow A = D(D^{-1}C) = DC$,where $(D^{-1}C) = C$

So we have a matrix A equal to a certain square matrix D multiplied by other matrix C , or we can write

$A = (AD^{-1})D = ED$ where in this case D is an $n \times n$ matrix, since multiplication of matrices is associative.

Now suppose we have an $m \times n$ matrix A which is not square and we we want to decompose this matrix into a product of a certain $m \times k$ matrix w and other matrix , in this case we can form an invertible $m \times m$ matrix D by multiplying w by any $k \times m$ matrix h(i.e D=wh)

By this method we can factorize any matrix to a product of any matrices we want and this can be done by two ways either a) we decompose the main matrix into two matrices and then we decompose each one of these two matrices using this easy method to get four matrices, then again we can decompose each one of these four matrices by the same way, and so on then by using the association law of matrix multiplication we get the result (at every step of decomposition we use different invertible matrices) , or

b) by writing $A = AD_1D_1^{-1}...D_n^{-1}D_n$ where D_i is invertible $n \times n$ matrix and using association law to get $A = (AD_1)(D_1^{-1}D_2)..(D_{n-1}D_n^{-1})D_n$ or any other combination Theorem (2)

Every $m \times n$ matrix A can be written as A=BC where B and C are two matrices of sizes $m \times k$ and $k \times n$ respectively where $m \le k$

Proof

We can form a none singular matrix D=WH where W is of type $m \times k$ and H of type $k \times m$

Now we have an invertible square matrix D of type $m \times m$

So we can write

$DA = WHA$ (!)

Multiplying both sides with D^{-1} we get

$D^{-1}DA = D^{-1}WHA$

$\Rightarrow A = (D^{-1}W)(HA)$ (2)

So by taking $B = (D^{-1}W)$, $C = (HA)$ theorem (2) will be proved.

Now in equation (2) we have

$A = (D^{-1}W)(HA)$ $*$

so substituting for the last A in $*$ by $A = (D^{-1}W)(HA)$, then using associative law of matrix multiplication , we get

$A = (D^{-1}W)H(D^{-1}WH)A$

Again we can substitute for the last A to get

$A = (D^{-1}W)H(D^{-1}W)H(D^{-1}W)HA$, and so on

$$A = (D^{-1}W)H(D^{-1}W)H...(D^{-1}W)HA = EHEH...EHA \quad (3)$$

where $E = (D^{-1}W)$ and obviously $(D^{-1}W)H = I_m$,

where I_m is the identity matrix of rank m.

Since we have $I_m = I_m \times I_m ... \times I_m$ we can deduce that

$$I_m = (D^{-1}W)H(D^{-1}W)H...(D^{-1}W)H = EHEH...EH \quad (4)$$

So we can also analyze the identity matrix of any rank using this method.

The benefit of the easy approach is that ,if a matrix can be decompose to certain matrices by using this approach we can get the other factors ,to show this suppose we have a matrix

$T = \begin{bmatrix} 1 & 3 & 1 \\ 4 & 2 & 1 \end{bmatrix}$ and we are given two matrices .

$A = \begin{bmatrix} 1 & 0 & 2 \\ 4 & 2 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 4 & 2 \\ 1 & 2 \end{bmatrix}$,

We want to find a matrix C such that $T = ABC$.

We can do this as follows:

$$AB = \begin{bmatrix} 1 & 0 & 2 \\ 4 & 2 & 5 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 4 & 2 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 17 & 14 \end{bmatrix} = D$$

$$D^{-1} = \begin{bmatrix} \frac{-7}{13} & \frac{2}{13} \\ \frac{17}{26} & \frac{-3}{26} \end{bmatrix}$$

Next we have

$$T = DD^{-1}T = ABD^{-1}T$$

$$= \begin{bmatrix} 1 & 0 & 2 \\ 4 & 2 & 5 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 4 & 2 \\ 1 & 2 \end{bmatrix} \left(\begin{bmatrix} \frac{-7}{13} & \frac{2}{13} \\ \frac{17}{26} & \frac{-3}{26} \end{bmatrix} \begin{bmatrix} 1 & 3 & 1 \\ 4 & 2 & 1 \end{bmatrix} \right)$$

$$\Rightarrow T = \begin{bmatrix} 1 & 0 & 2 \\ 4 & 2 & 5 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 4 & 2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} \frac{1}{13} & \frac{-17}{13} & \frac{-5}{13} \\ \frac{5}{26} & \frac{45}{26} & \frac{7}{13} \end{bmatrix}$$

$$\Rightarrow C = \begin{bmatrix} \frac{1}{13} & \frac{-17}{13} & \frac{-5}{13} \\ \frac{5}{26} & \frac{45}{26} & \frac{7}{13} \end{bmatrix}$$

Example .1

Given $A = \begin{bmatrix} 2 & 3 & 4 & 1 & 1 \\ 5 & 0 & 1 & 2 & 1 \end{bmatrix}$, write A= BC where B is 2×4 matrix and C is 4×5 one.

Solution By taking

$$W = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{bmatrix}, H = \begin{bmatrix} -1 & -2 \\ 1 & 1 \\ 1 & 0 \\ 0 & 5 \end{bmatrix}$$

$$\Rightarrow D = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & -2 \\ 1 & 1 \\ 1 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$$

$$\Rightarrow D^{-1} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

Now we have

$$DA = \left(\begin{bmatrix} 1 & 1 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & -2 \\ 1 & 1 \\ 1 & 0 \\ 0 & 5 \end{bmatrix} \right) \begin{bmatrix} 2 & 3 & 4 & 1 & 1 \\ 5 & 0 & 1 & 2 & 1 \end{bmatrix}$$

Multiplying both sides with D^{-1} we get

$$D^{-1}DA = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \left(\begin{bmatrix} 1 & 1 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & -2 \\ 1 & 1 \\ 1 & 0 \\ 0 & 5 \end{bmatrix} \right) \begin{bmatrix} 2 & 3 & 4 & 1 & 1 \\ 5 & 0 & 1 & 2 & 1 \end{bmatrix}$$

$$\Rightarrow A = \left(\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{bmatrix} \right) \left(\begin{bmatrix} -1 & -2 \\ 1 & 1 \\ 1 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} 2 & 3 & 4 & 1 & 1 \\ 5 & 0 & 1 & 2 & 1 \end{bmatrix} \right)$$

$$\Rightarrow A = \begin{bmatrix} 4 & 3 & 2 & 1 \\ 3 & 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} -12 & -3 & -6 & -5 & -3 \\ 7 & 3 & 5 & 3 & 2 \\ 2 & 3 & 4 & 1 & 1 \\ 25 & 0 & 5 & 10 & 5 \end{bmatrix}$$

Note

Since matrix factorization is not unique ,we can use more simple matrices W, H, D, D^{-1}

Also for writing any 2×5 matrix A, as A=BC where B is 2×4 matrix and C is 4×5 one we can use these four matrices

$$W = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{bmatrix}, H = \begin{bmatrix} -1 & -2 \\ 1 & 1 \\ 1 & 0 \\ 0 & 5 \end{bmatrix}, D = WH = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix},$$

$$, D^{-1} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

call these four matrices standard matrices , or we can use any other four matrices satisfying the conditions, in general for every matrix A of type $m \times n$ we can write A=BC where B is $m \times k$ matrix and C is $k \times n$ one such that $k \geq m$ by taking four standard matrices W,H,D, D^{-1} where W is an $m \times k$ matrix and H is $k \times n$ matrix.

IV. CONCLUSION .

This paper propose very effective and simple method for matrix decomposition by using invertible matrices ,so by easy method we can factorize any matrix to certain matrices we want and other matrices ,although there are many methods for matrix factorization ,this easy method will be the simplest one of them. Unlike the other approach we can decompose any matrix to any number of factors (matrices). Easy approach has the benefit that for every $m \times n$ matrix A we can take four standard matrices W,H,D, D^{-1} where W is an $m \times k$ matrix, H is $k \times n$ to factorize A.

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