The Sine-Cosine Function Method for the Exact Solution of the Classical Boussinesq (CB) and the Mikhailov-Shabat (MS) Equations

Syeda Naima Hassan, Anwar Ja'afar Mohamad Jawad

***Abstract*— In this paper, we establish traveling wave solutions by using sine-cosine function method for Classical Boussinesq (CB), and the Mikhailov-Shabat (MS) equations which are nonlinear partial differential equations and also important soliton equations. It is shown that the sine-cosine method provides a powerful mathematical tool for solving many nonlinear partial differential equations in mathematical physics.**

***Index Terms*— Sine-Cosine function method, Traveling wave equation, Classical Boussinesq equation, Mikhailov-Shabat equations.**

I. INTRODUCTION

The nonlinear evolution equations have a major role in various scientific and engineering fields, such as fluid mechanics, plasma physics, optical fibers, solid state physics, chemical physics and geochemistry. Nonlinear wave phenomena of dispersion, dissipation, diffusion, reaction and convection are very important in nonlinear wave equations. In recent years, quite a few methods for obtaining explicit traveling and solitary wave solutions of nonlinear evolution equations have been proposed. A variety of powerful methods, such as, tanh sech method {Malfliet W. [1], Khater et al. [2], and Wazwaz [3]}, extended tanh method {El-Wakil et al. [4], Fan [5], Wazwaz [6]}, hyperbolic function method {Xia and Zhang[7], and Yusufoglu and Bekir [8]}, Jacobi elliptic function expansion method {Inc and Ergut[9]}, F-expansion method {Zhang [10]}, and the First Integral method {Feng [11], Ding and Li [12]} . The sine-cosine method {Mitchell [13], Parkes [14], and Khater [2]} has been used to solve different types of nonlinear systems of PDEs.

The aim of this paper is to find the exact and periodic solutions by the sine-cosine method of the Mikhailov-Shabat (MS) equation and Classical Boussinesq (CB) equation, which are the important soliton equations.

II. THE SINE-COSINE FUNCTION METHOD

We introduce the wave variable $\xi = x - ct$ into the PDE

$$p(u, u_t, u_x, u_{tt}, u_{xx}, u_{tx}, \dots\dots\dots) = 0. \qquad (1)$$

Where u(x,t) is traveling wave solution. This enables us to use the following changes:

$$\frac{\partial}{\partial t} = -c\frac{\partial}{\partial \xi}, \frac{\partial^2}{\partial t^2} = c^2\frac{\partial^2}{\partial \xi^2}, \frac{\partial}{\partial x} = \frac{\partial}{\partial \xi}, \frac{\partial^2}{\partial x^2} = \frac{\partial^2}{\partial \xi^2} \qquad (2)$$

One can immediately reduce the nonlinear PDE (1) into a nonlinear ODE

$$Q(u, u_\xi, u_{\xi\xi}, u_{\xi\xi\xi}, \dots\dots\dots) = 0 \qquad (3)$$

The ordinary differential equation (3) is then integrated as long as all terms contain derivatives, where we neglect integration constants.

The solutions of many nonlinear equations can be expressed in the form {Ali et al. [15], Wazwaz [16]-[17]}

$$u(\xi) = \begin{cases} \lambda \sin^\beta(\mu\xi), & |\xi| \le \frac{\pi}{\mu}, \\ 0 & ,otherwise \end{cases} \qquad (4)$$

Or, in the form

$$u(\xi) = \begin{cases} \lambda \cos^\beta(\mu\xi), & |\xi| \le \frac{\pi}{\mu}, \\ 0 & ,otherwise \end{cases} \qquad (5)$$

Where λ, μ and β≠0 are parameters that will be determined, μ and c are the wave number and the wave speed respectively. We use

$$u(\xi) = \lambda \sin^\beta(\mu\xi)$$
$$u^n = \lambda^n \sin^{n\beta}(\mu\xi)$$
$$(u^n)_\xi = n\mu\beta\lambda^n \cos(\mu\xi)\sin^{n\beta-1}(\mu\xi) \qquad (6)$$
$$(u^n)_{\xi\xi} = -n^2\mu^2\beta^2\lambda^n \sin^{n\beta}(\mu\xi) + n\mu^2\lambda^n\beta(n\beta-1)\sin^{n\beta-2}(\mu\xi)$$

And

$$u(\xi) = \lambda \cos^\beta(\mu\xi)$$
$$u^n = \lambda^n \cos^{n\beta}(\mu\xi)$$
$$(u^n)_\xi = -n\mu\beta\lambda^n \sin(\mu\xi)\cos^{n\beta-1}(\mu\xi) \qquad (7)$$
$$(u^n)_{\xi\xi} = -n^2\mu^2\beta^2\lambda^n \cos^{n\beta}(\mu\xi) + n\mu^2\lambda^n\beta(n\beta-1)\cos^{n\beta-2}(\mu\xi)$$

And so on for other derivatives.

We substitute (6) or (7) into the reduced equation obtained above in (3), balance the terms of the cosine functions when (7) is used, or balance the terms of the sine functions when (6) is used, and solving the resulting system of algebraic equations by using the computerized symbolic calculations We next collect all terms with same power in $\cos^k(\mu\xi)$ or $\sin^k(\mu\xi)$ and set to zero their coefficients to get a system of algebraic equations' among the unknowns μ, β and λ. We obtained all possible value of the parameters μ, β, and λ.

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III. APPLICATIONS

A. The Classical Boussinesq Equations

The Classical Boussinesq (CB) equations

$$u_t + [(1+u)v]_x = -\frac{1}{4}v_{xxx}$$
$$v_t + vv_x + u_x = 0 \qquad (8)$$

In order to obtain travelling wave solutions of equation (8), we make the transformations

$u(x,t) = u(\xi);\ v(x,t) = v(\xi);\ \xi = x - wt$

And integrating once with respect to ξ, then (8) becomes;

$$-wu + (1+u)v + \frac{1}{4}v'' = 0 \qquad (9)$$

And

$$u = wv - \frac{1}{2}v^2 \qquad (10)$$

Substituting Eq. (10) into Eq. (9) leads to the following ODE

$$v'' + 4(1-w^2)v + 6wv^2 - 2v^3 = 0 \qquad (11)$$

Using (6) into (11) we get,

$$-\mu^2\beta^2\lambda\sin^{\beta}(\mu\xi) + \mu^2\lambda\beta(\beta-1)\sin^{\beta-2}(\mu\xi) + 4(1-w^2)\lambda\sin^{\beta}(\mu\xi) + 6w\lambda^2\sin^{2\beta}(\mu\xi) - -2\lambda^3\sin^{3\beta}(\mu\xi) = 0 \qquad (12)$$

Equating the exponents and the coefficients of each pair of the sine functions we find the following algebraic system:

$$\beta(\beta-1) \neq 0$$
$$\beta - 2 = 3\beta \rightarrow \beta = -1 \qquad (13)$$

Substituting Eq. (13) into Eq. (12) to get:

$$-\mu^2\lambda\sin^{-1}(\mu\xi) + 2\mu^2\lambda sin^{-3}(\mu\xi) + 4(1-w^2)\lambda\sin^{-1}(\mu\xi) + 6w\lambda^2\sin^{-2}(\mu\xi) - 2\lambda^3\sin^{-3}(\mu\xi) = 0 \qquad (14)$$

Equating the exponents and the coefficients of each pair of the sine function, we obtain a system of algebraic equations:

$$sin^{-3}(\mu\xi):\ 2\mu^2\lambda - 2\lambda^3 = 0$$
$$sin^{-2}(\mu\xi):\ 6w\lambda^2 = 0 \qquad (15)$$
$$\sin^{-1}(\mu\xi):\ -\mu^2\lambda + 4(1-w^2)\lambda = 0$$

By solving the algebraic system (15), we get

$$\beta = -1;\ \mu = \pm 2\sqrt{(1-w^2)}\ ;\ \lambda = \pm 2\sqrt{(1-w^2)} \qquad (16)$$

In view of (4) and (16) we obtain the periodic solutions

$$v = \pm 2\sqrt{(1-w^2)}\ \sin^{-1}\left(\pm 2\sqrt{(1-w^2)}\ \xi\right) \qquad (17)$$

Where $\xi = x - wt$.

To find the solutions $u(x,t)$, according to (10)

$$u = wv - \frac{1}{2}v^2 \qquad (18)$$

By means of the equations (4) and (16) and using equation (18), we have the following periodic solutions for $u(x,t)$:

$$u = \pm 2\sqrt{(1-w^2)}\ w\ \sin^{-1}(\pm 2\sqrt{(1-w^2)}\ \xi) - 2(1-w^2)\{\sin^{-1}(\mu\xi)\}^2 \qquad (19)$$

Where $\xi = x - wt$.

From the equations (17) and (19) we get the solutions of the CB *equation* by the *sine-cosine* method.

B. The Mikhailov-Shabat (MS) Equation

In this section we deal with the Mikhailov-Shabat (MS) equations

$$p_t = p_{xx} + (p+q)q_x - \frac{1}{6}(p+q)^3$$
$$-q_t = q_{xx} - (p+q)p_x - \frac{1}{6}(p+q)^3 \qquad (20)$$

In order to solve MS system (20) we now introduce the transformation

$$u(x,t) = p(x,t) + q(x,t)$$
$$v(x,t) = q_x(x,t) - p_x(x,t) \qquad (21)$$

Then the MS system (20) becomes

$$u_t + v_x - uu_x = 0$$
$$v_t + (uv)_x - u^2u_x + u_{xxx} = 0 \qquad (22)$$

Substituting

$$u(x,t) = u(\xi), v(x,t) = v(\xi), \xi = k(x-ct) \qquad (23)$$

(Where k and c are real constants) into (22) and integrating once with respect to ξ we get

$$v = cu + \frac{1}{2}u^2 \qquad (24)$$

And

$$k^2u'' + uv - \frac{1}{3}u^3 - cv = 0 \qquad (25)$$

Taking (24) into (25), we obtain the following ODE;

$$u'' + \frac{c}{2k^2}u^2 - \frac{c^2}{k^2}u + \frac{1}{6k^2}u^3 = 0 \qquad (26)$$

Seeking solutions of the form (6) we get:

$$-\mu^2\beta^2\lambda\sin^{\beta}(\mu\xi) + \mu^2\lambda\beta(\beta-1)\sin^{\beta-2}(\mu\xi) + \frac{c}{2k^2}\lambda^2\sin^{2\beta}(\mu\xi) - \frac{c^2}{k^2}\lambda\sin^{\beta}(\mu\xi) + \frac{1}{6k^2}\lambda^3\sin^{3\beta}(\mu\xi) = 0 \qquad (27)$$

Equating the exponents and the coefficients of each pair of the sine functions we find the following algebraic system:

$$\beta(\beta-1) \neq 0$$
$$\beta - 2 = 3\beta \rightarrow \beta = -1 \qquad (28)$$

Substituting Eq. (28) into Eq. (27) to get:

$$-\mu^2\lambda \sin^{-1}(\mu\xi) + 2\mu^2 \lambda\sin^{-3}(\mu\xi) + \frac{c}{2k^2}\lambda^2 \sin^{-2}(\mu\xi) - \frac{c^2}{k^2}\lambda\sin^{-1}(\mu\xi) + \frac{1}{6k^2}\lambda^3 \sin^{-3}(\mu\xi) = 0$$

Equating the exponents and the coefficients of each pair of the sine function, we obtain a system of algebraic equations:

$$\begin{aligned} &\sin^{-3}(\mu\xi): 2\mu^2\lambda + \frac{1}{6k^2}\lambda^3 = 0 \\ &\sin^{-2}(\mu\xi): \frac{c}{2k^2}\lambda^2 = 0 \\ &\sin^{-1}(\mu\xi): -\mu^2\lambda - \frac{c^2}{k^2}\lambda = 0 \end{aligned} \tag{29}$$

By solving the algebraic system (29), we get, when $\frac{c^2}{k^2} < 0$,

$$\beta = -1\,; \mu = \pm\sqrt{12}\ c\,;\ \lambda = \pm\sqrt{12}\ c \tag{30}$$

In view of (4), (5), (23) and (30), for $\frac{c^2}{k^2} < 0$, we obtain the periodic solutions

$$u(x,t) = \pm\sqrt{12}\ c\csc[\pm\sqrt{-\frac{c^2}{k^2}}\ k(x-ct)] \tag{31}$$

Where, $0< \pm\sqrt{-\frac{c^2}{k^2}}\ k(x-ct) <\pi$

And

$$u(x,t) = \pm\sqrt{12}\ c\sec[\pm\sqrt{-\frac{c^2}{k^2}}\ k(x-ct)] \tag{32}$$

Where, $0< \pm\sqrt{-\frac{c^2}{k^2}}\ k(x-ct) <\frac{\pi}{2}$

And for $\frac{c^2}{k^2} > 0$, we obtain the periodic solutions

$$u(x,t) = \pm i\sqrt{12}\, c\, csc\,h[\pm\sqrt{\frac{c^2}{k^2}}\ k(x-ct)] \tag{33}$$

And

$$u(x,t) = \pm\sqrt{12}\ c\,\text{sech}[\pm\sqrt{\frac{c^2}{k^2}}\ k(x-ct)] \tag{34}$$

To find the solutions $v(x,t)$, according to (24)

$$v = cu + \frac{1}{2}u^2 \tag{35}$$

By means of the equations (4), (5) and (23) and using equation (35), we have the following periodic solutions for $v(x,t)$:

When $\frac{c^2}{k^2} < 0$, we get

$$v(x,t) = \pm\sqrt{12}\ c^2\csc[\pm\sqrt{-\frac{c^2}{k^2}}\ k(x-ct)] + 6\,c^2\csc^2[\pm\sqrt{-\frac{c^2}{k^2}}\ k(x-ct)] \tag{36}$$

And

$$v(x,t) = \pm\sqrt{12}\ c^2\sec[\pm\sqrt{-\frac{c^2}{k^2}}\ k(x-ct)] + 6\,c^2\sec^2[\pm\sqrt{-\frac{c^2}{k^2}}\ k(x-ct)] \tag{37}$$

When $\frac{c^2}{k^2} > 0$, we get

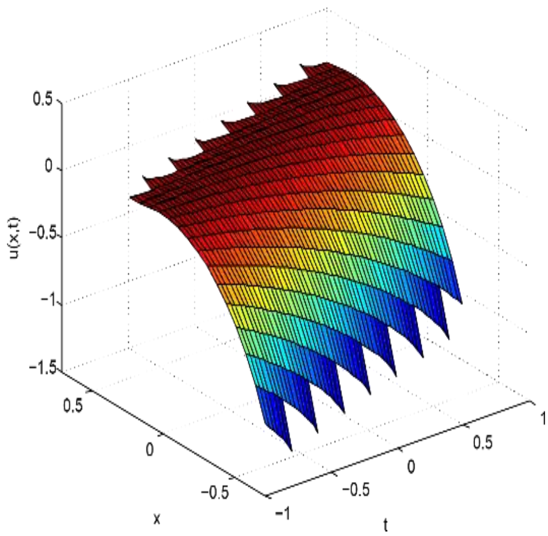
$$v(x,t) = \pm i\sqrt{12}\ c^2\,\text{csch}\,[\pm\sqrt{\frac{c^2}{k^2}}\ k(x-ct)] - 6\,c^2\,\text{csch}^2\left[\pm\sqrt{\frac{c^2}{k^2}}\ k(x-ct)\right] \tag{38}$$

And

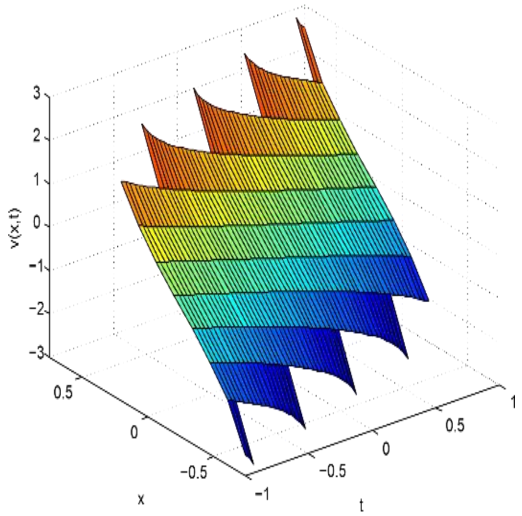
$$v(x,t) = \pm\sqrt{12}\ c^2\,\text{sech}[\pm\sqrt{\frac{c^2}{k^2}}\ k(x-ct)] + 6\,c^2\,\text{sech}^2[\pm\sqrt{\frac{c^2}{k^2}}\ k(x-ct)] \tag{39}$$

IV. GRAPHICAL INTERPRETATION

In this section, we will put forth the graphical representation of determined traveling wave solutions of the Classical Boussinesq (CB) equations.

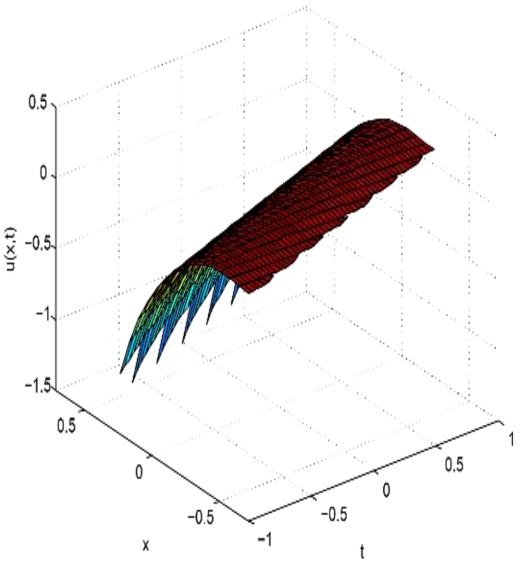


1(a)

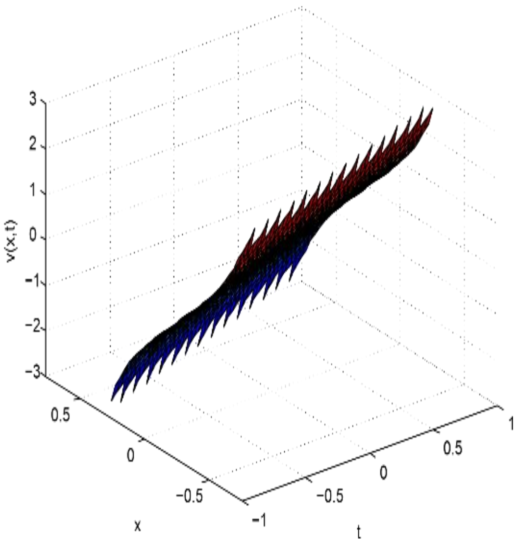


1(b)

Figure 1: (a) u(x,t) in equations (19) and (b) v(x,t) in equations (17) where $w = 0.25, \mu = 1$ & taking +ve sign.

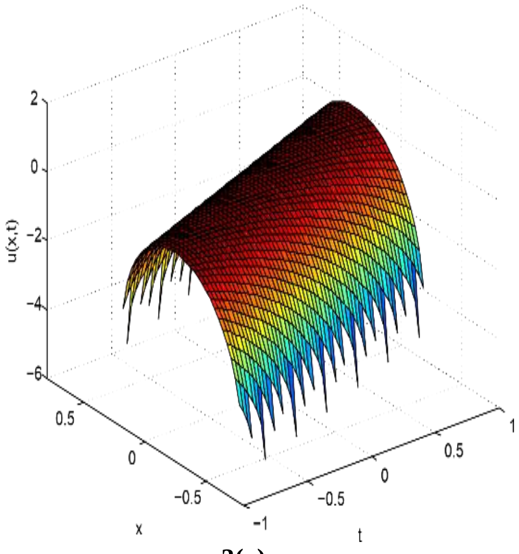


2(a)

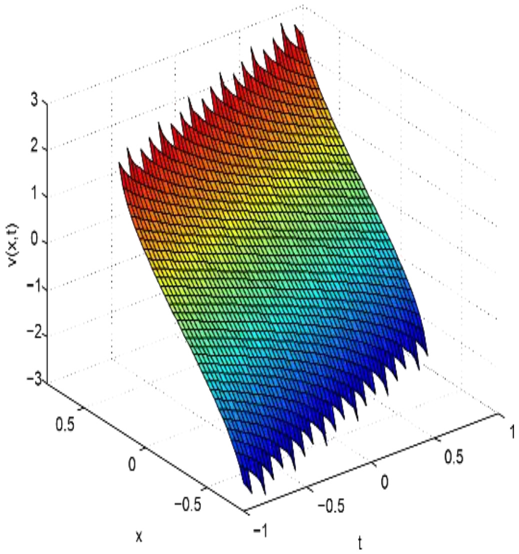


2(b)

Figure 2:(a) u(x,t) in equations (19) and (b) v(x,t) in equations (17) where $w = 0.25, \mu = 1$ & taking -ve sign.

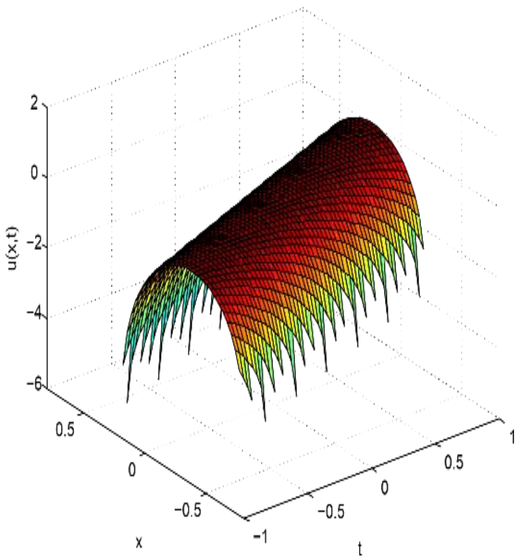


3(a)

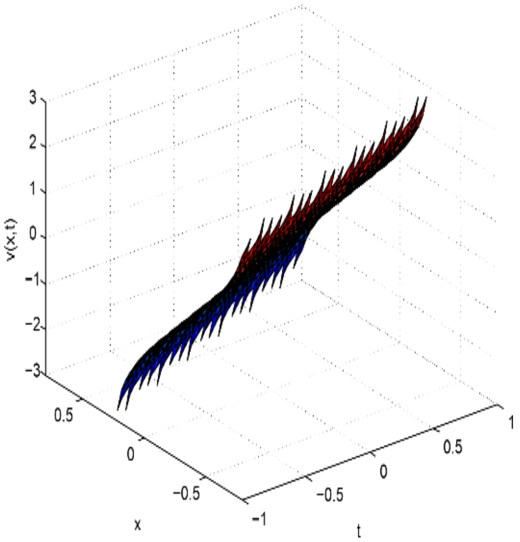


3(b)

Figure 3: (a) u(x,t) in equations (19) and (b) v(x,t) in equations (17) where $w = 0.3, \mu = 2$ & taking +ve sign

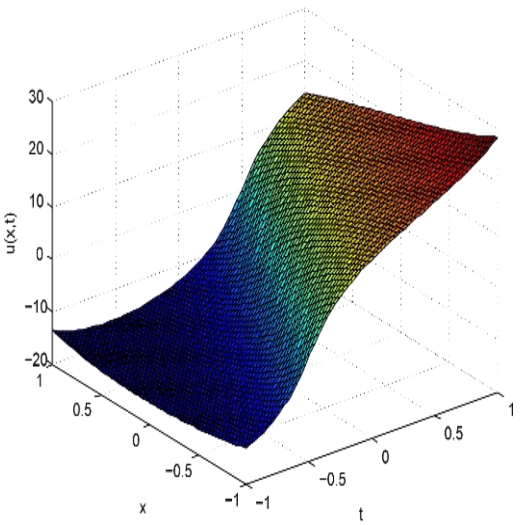


4(a)

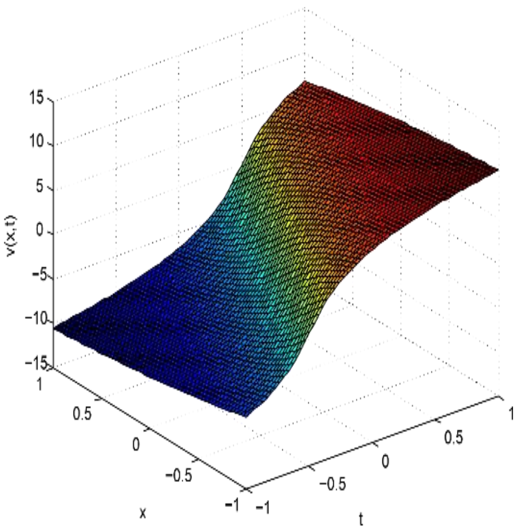


4(b)

Figure 4: (a) u(x,t) in equations (19) and (b) v(x,t) in equations (17) where $w = 0.3, \mu = 2$ & taking -ve sign.

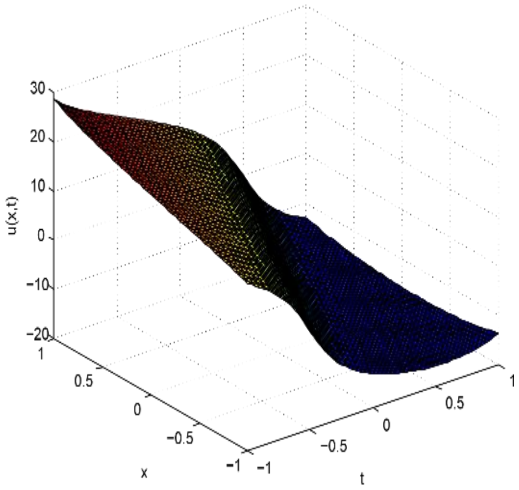


5(a)

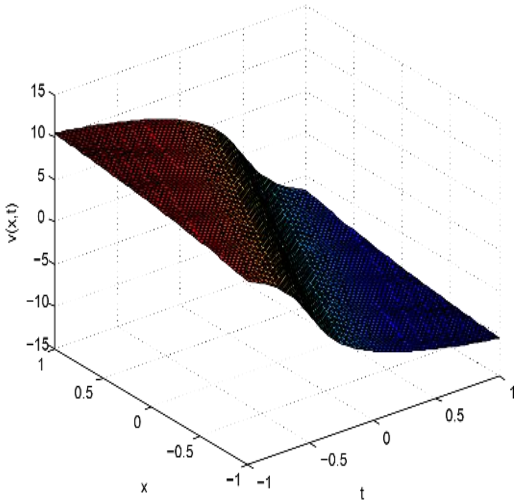


5(b)

Figure 5:(a) u(x,t) in equations (19) and (b) v(x,t) in equations (17) where $w = 2, \mu = 0.3$ & taking +ve sign.



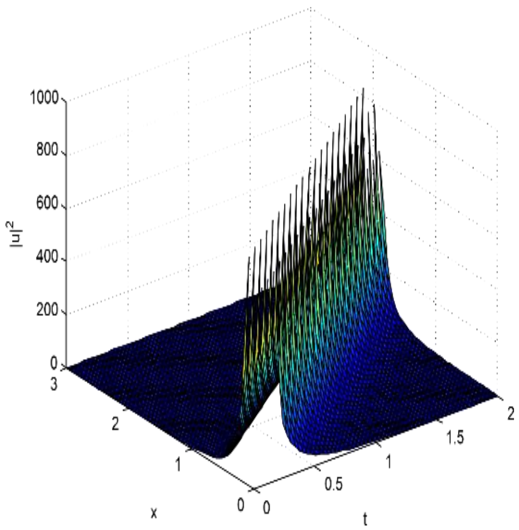
6(a)



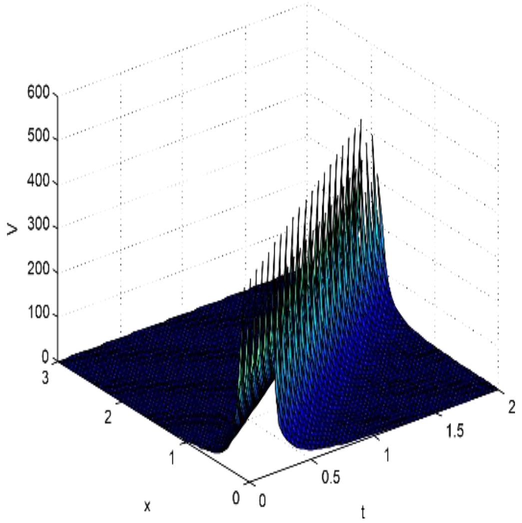
6(b)

Figure 6:(a) u(x,t) in equations (19) and (b) v(x,t) in equations (17) where $w = 2, \mu = 0.3$ & taking -ve sign.

In this section, we will put forth the graphical representation of determined traveling wave solutions of Mikhailov-Shabat (MS) equation

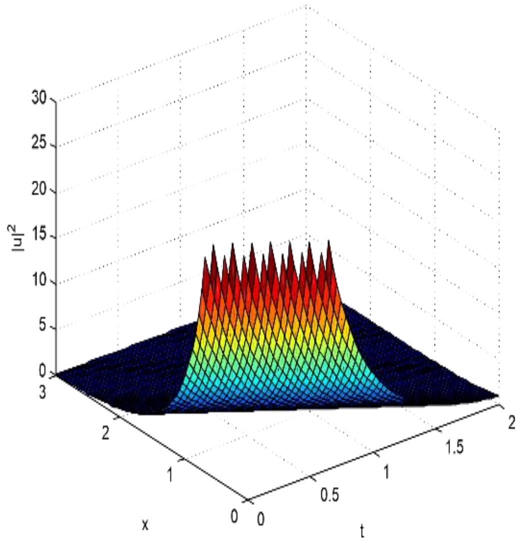


7(a)

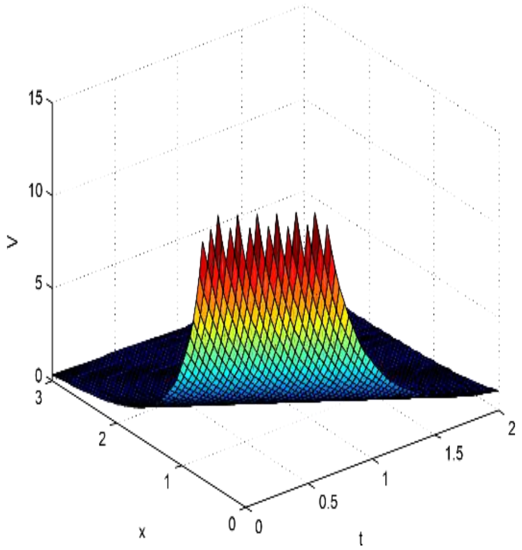


7(b)

Figure 7: (a) u(x,t) in equations (31) and (b) v(x,t) in equations (36) where $k = 0.25, c = 1$.

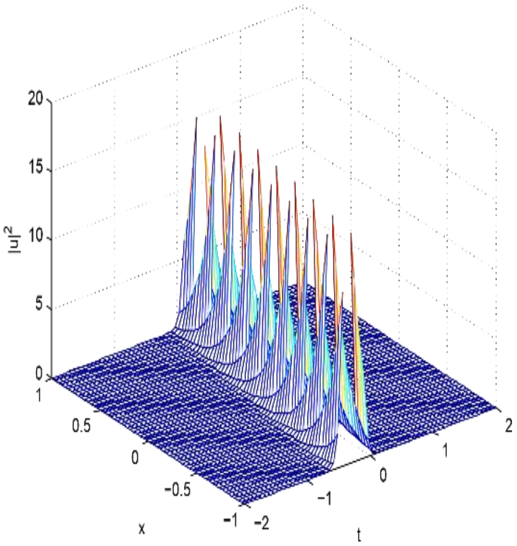


8(a)

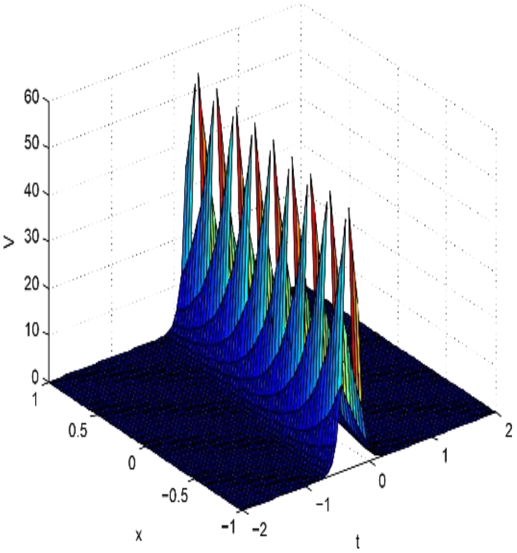


8(b)

Figure 8: (a) u(x,t) in equations (31) and (b) v(x,t) in equations (36) where $k = 0.25, c = -1$.

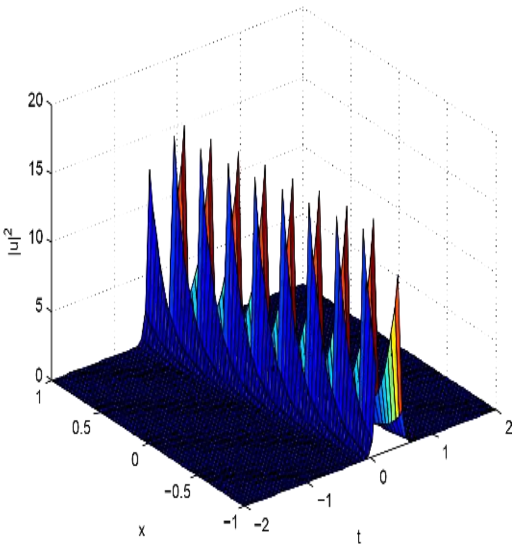


9(a)

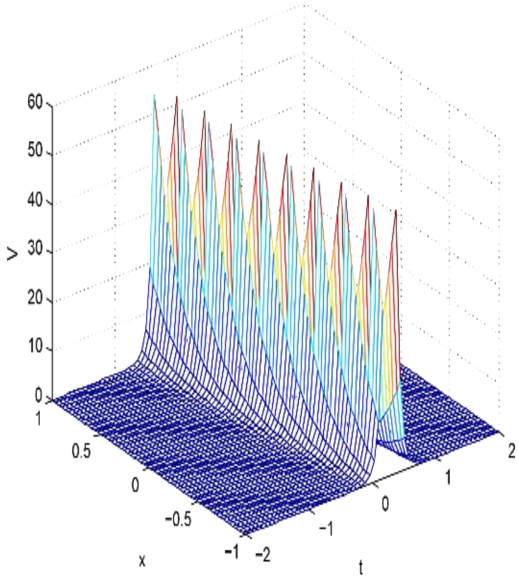


9(b)

Figure 9: (a) u(x,t) in equations (32) and (b) v(x,t) in equations (37) where $k = 1.5, c = 3.5$

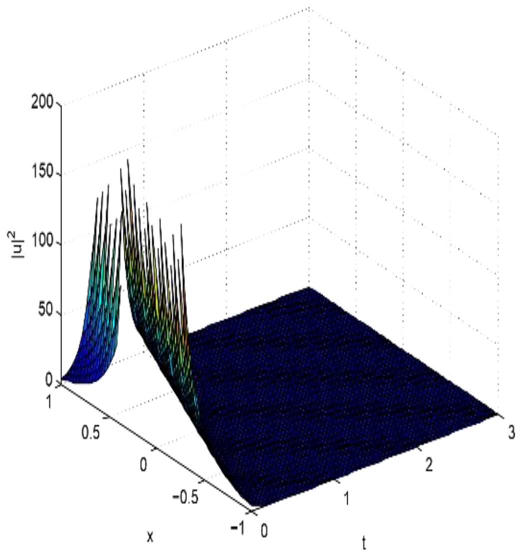


10(a)

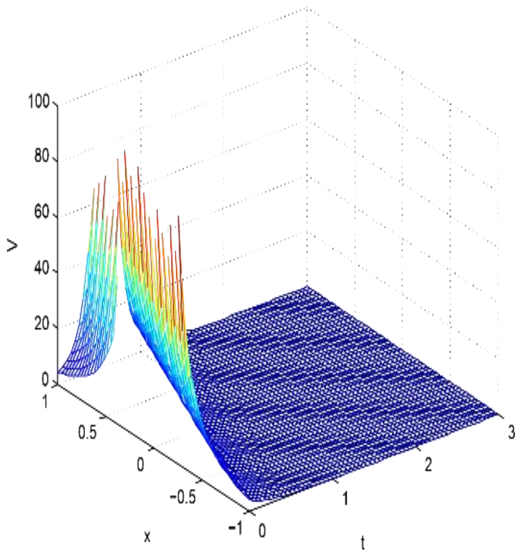


10(b)

Figure 10: (a) u(x,t) in equations (32) and (b) v(x,t) in equations (37) where $k = 1.5, c = -3.5$

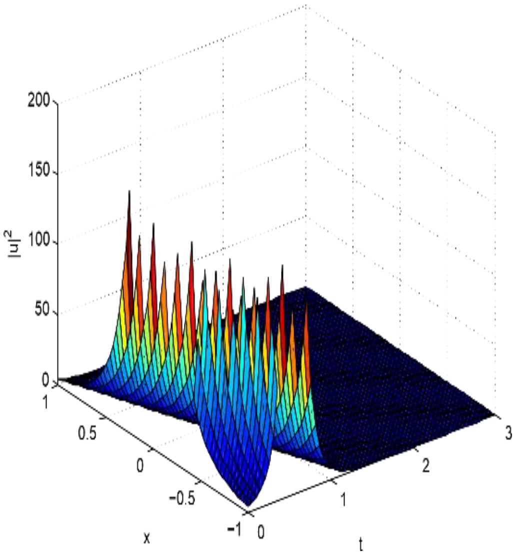


11(a)

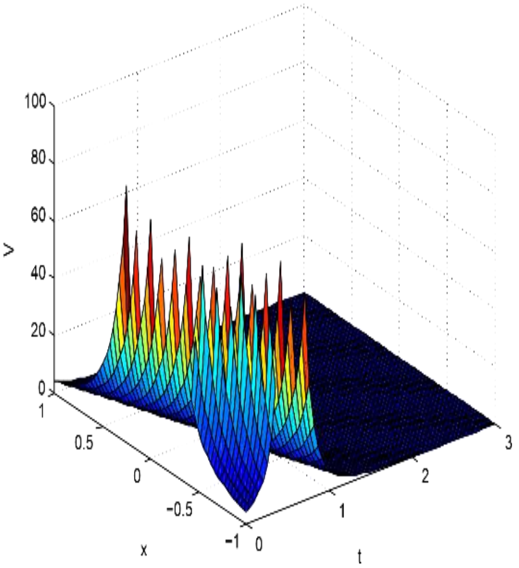


11(b)

Figure 11: (a) u(x,t) in equations (33) and (b) v(x,t) in equations (38) where $k = 2.5, c = 1.8$

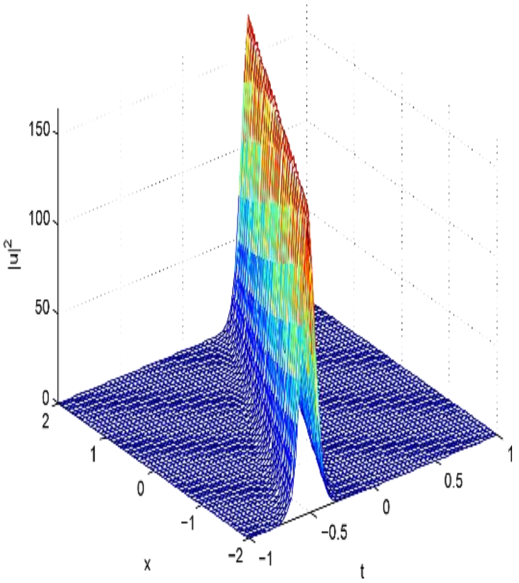


12(a)

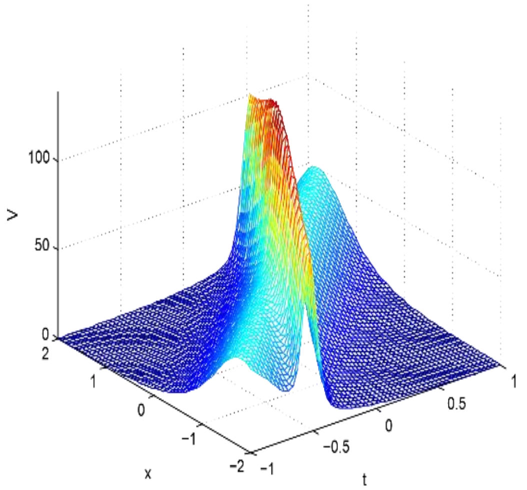


12(b)

Figure 12: (a) u(x,t) in equations (33) and (b) v(x,t) in equations (38) where $k = -2.5, c = -1.8$

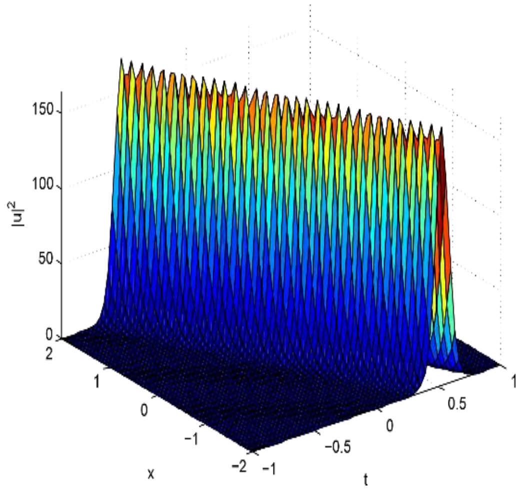


13(a)

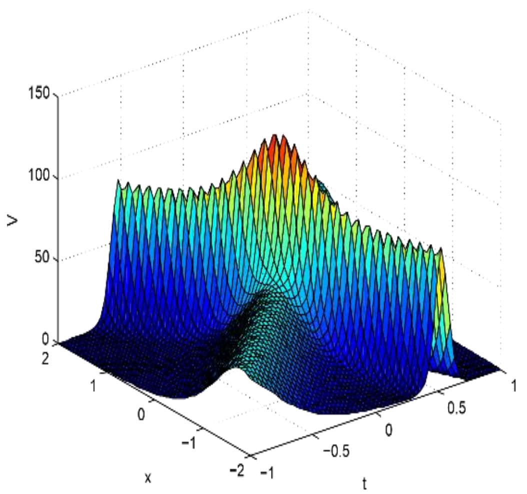


13(b)

Figure 13:(a)u(x,t) in equations (34) and (b) v(x,t) in equations (39) where $k = 4.2, c = 3.8$



14(a)



14(b)

Figure 14:(a) u(x,t) in equations (34) and (b) v(x,t) in equations (39) where $k = 4.2, c = -3.8$

V. CONCLUSION

In this paper, we proposed the *sine-cosine* method to seek the periodic solutions of the NPDE. It is shown that this method is more powerful in giving more kinds of solutions. By making use of the method, we study the Mikhailov-Shabat (MS) equations' and the classical system of Boussinesq equations', new families of solutions written are found. The method is used to find a new exact and periodic solutions for the CB and MS equations. Thus, we can say that the proposed method can be extended to solve the problems of nonlinear partial differential equations which arising in the theory of solitons and other areas. These solution was similar to the solutions obtained in other paper. The study reveals the power of the method.

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