A simple algorithm for Optimal Control problems governed by Non-linear Hammerstein Integral Equations

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***Abstract*— In this study, a novel computational framework for solving a class of optimal control problem for systems governed by non-linear Hammerstein integral equations is presented. A theorem for the convergence and the validity of the approach is also given in detail. Numerical experiments and comparisons with exact solutions confirm the efficiency and the accuracy of the proposed technique.**

***Index Terms*— Optimal Control problem, Non-linear Hammerstein integral equations, Iterative methods, Approximate-analytical solution.**

I. INTRODUCTION

Iterative schemes and optimizations related to integral equations are two prominent fields of research in applied science and engineering. The major purpose of optimization is to determine procedures of how optimally change or influence real systems to achieve a desired result. This requires to realize large-scale optimization strategies with increasing complexity which in turn motivates the development of numerical techniques for optimization purposes.

On the other hand, in mathematical formulation of physical phenomena, integral equations are always encountered and have attracted much attention. In fact, integral equations are as important as differential equations and appear in a variety of applications in many fields including continuum mechanics, potential theory, geophysics, electricity and magnetism, kinetic theory of gases, hereditary phenomena in biology, quantum mechanics, mathematical economics, population genetics, medicine, fluid mechanics, steady state heat conduction, and radiative heat transfer problems [1, 3, 6, 7, 9, 10, 12, 13, 14, 21, 23, 25]. In this way there are many direct and indirect numerical solution for integral equations such as [2, 5, 17, 17, 22]. Furthermore, optimal control of systems governed by integral equations are momentous in applications such as the optimal control problem related to the Ornstein-Uhlenbeck process which arises from statistical communication theory [8]. In this paper, we focus on the formulation of a class of optimal control problems governed by non-linear Hammerstein integral equations as follows

$$Minimize\quad J(x,u)=\int_0^1 f_0(t,x(t),u(t))dt, \tag{1}$$

subject to

$$x(t)=y(t)+\lambda_1\int_0^t k_1(t,s,u(s))\Phi(s,x(s))ds \tag{2}$$

$$+\lambda_2\int_0^1 k_2(t,s,u(s))\Psi(s,x(s))ds,$$

where the known function $f_0(t,x(t),u(t))\in C([0,1]\times R\times R)$ and $y(t),\Phi(t,x(t)),\ \Psi(t,x(t)),k_1(t,s,u(s)),k_2(t,s,u(s))$ are given functions which are defined on the interval $0\le t,s\le 1$, and can be expanded to the Taylor series about $t=c, 0\le c\le 1$. Besides, $x(t),u(t)\in C^\infty([0,1])$ are the trajectory and control functions, respectively. Here, it is assumed that the problem (1)-(2) has a unique solution.

Nevertheless, there is no research to solve the optimal control problem (1)-(2). Thus, the main purpose of this study is to construct an iterative scheme to obtain the approximate solution and also the analytical solution of the problem in the form of polynomial series solution.

II. TAYLOR SOLUTION

In this section, following the work of Mahmoodi [11], it is assumed that for a given $u(s)$, the non-linear integral equation(2), for which $y(t),\Phi(t,x(t)),\Psi(t,x(t))$, $k_1(t,s,u(s))$ and $k_2(t,s,u(s))$ are functions that have suitable derivatives in the interval $0\le s,t\le 1$, has the solution $x(t)$ in the form

$$x(t)=\sum_{n=0}^{N}\frac{1}{n!}x^{(n)}(c)(t-c)^n,\quad 0\le t,c\le 1. \tag{3}$$

Noting that, Eq.(3) is a Taylor polynomial of degree N at $t=c$, where $x^{(n)}(c)$, $n=0,1,...,N$ are coefficients to be determined.

Using Eq.(3), we also consider $\Phi(t,x(t))$ and $\Psi(t,x(t))$ in Eq.(2) expressed in terms of Taylor polynomials as

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$$\Phi(t,x(t)) = \sum_{n=0}^{N} \frac{1}{n!}\phi_n (t-c)^n, \quad 0 \le t,c \le 1. \tag{4}$$

$$\Psi(t,x(t)) = \sum_{n=0}^{N} \frac{1}{n!}\psi_n (t-c)^n, \quad 0 \le t,c \le 1. \tag{5}$$

Which are Taylor polynomials of degree N at $t=c$. The coefficients $\phi_n, \psi_n, n = 0,1,...,N$ are non-linear combinations of $x^{(0)}(c), x^{(1)}(c),..., x^{(N)}(c)$ as follows:

$$\begin{aligned}
\phi_0 &= \Phi(t,x(t)) \\
\phi_1 &= \frac{\partial \Phi}{\partial t} + x'(t)\frac{\partial \Phi}{\partial x} \\
\phi_2 &= \frac{\partial^2 \Phi}{\partial t^2} + x'(t)\frac{\partial \Phi}{\partial x}\frac{\partial \Phi}{\partial t} + (x'(t))^2\frac{\partial^2 \Phi}{\partial x^2} + x''(t)\frac{\partial \Phi}{\partial x} \\
\phi_3 &= \frac{\partial^3 \Phi}{\partial t^3} + 2x''(t)\frac{\partial \Phi}{\partial x}\frac{\partial \Phi}{\partial t} + 2(x'(t))^2\frac{\partial^2 \Phi}{\partial x^2}\frac{\partial \Phi}{\partial t} + \\
& 3x''(t)x'(t)\frac{\partial^2 \Phi}{\partial x^2} + x'(t)\frac{\partial \Phi}{\partial x}\frac{\partial^2 \Phi}{\partial t^2} + (x'(t))^3\frac{\partial^3 \Phi}{\partial x^3} \\
& + x'''(t)\frac{\partial \Phi}{\partial x} \\
& \vdots
\end{aligned} \tag{6}$$

and

$$\begin{aligned}
\psi_0 &= \Psi(t,x(t)) \\
\psi_1 &= \frac{\partial \Psi}{\partial t} + x'(t)\frac{\partial \Psi}{\partial x} \\
\psi_2 &= \frac{\partial^2 \Psi}{\partial t^2} + x'(t)\frac{\partial \Psi}{\partial x}\frac{\partial \Psi}{\partial t} + (x'(t))^2\frac{\partial^2 \Psi}{\partial x^2} \\
& + x''(t)\frac{\partial \Psi}{\partial x} \\
\psi_3 &= \frac{\partial^3 \Psi}{\partial t^3} + 2x''(t)\frac{\partial \Psi}{\partial x}\frac{\partial \Psi}{\partial t} + 2(x'(t))^2\frac{\partial^2 \Psi}{\partial x^2}\frac{\partial \Psi}{\partial t} \\
& + 3x''(t)x'(t)\frac{\partial^2 \Psi}{\partial x^2} + x'(t)\frac{\partial \Psi}{\partial x}\frac{\partial^2 \Psi}{\partial t^@} \\
& + (x'(t))^3\frac{\partial^3 \Psi}{\partial x^3} + x'''(t)\frac{\partial \Psi}{\partial x} \\
& \vdots
\end{aligned} \tag{7}$$

A. Matrix representation of the components

To obtain the solution of (2) in the form of expression (3), we first differentiate it n times with respect to t:

$$x^{(n)}(t) = y^{(n)}(t) + \lambda_1 \frac{d^n}{dt^n}\int_0^t k_1(t,s,u(s))\Phi(s,x(s))ds$$
$$+ \lambda_2 \int_0^1 \frac{\partial^n k_2(t,s,u(s))}{\partial t^n}\Psi(s,x(s))ds \tag{8}$$

and then analyse it as matrix representation.

According to [15], Eq.(8) can be written as

$$x^{(n)}(t) = y^{(n)}(t) + \lambda_1 V^{(n)}(t) + \lambda_2 F^{(n)}(t) \tag{9}$$

where

$$V^{(n)}(t) = \frac{d^n}{dt^n}\int_0^t k_1(t,s,u(s))\Phi(s,x(s))ds$$

and

$$F^{(n)}(t) = \int_0^1 \frac{\partial^n k_2(t,s,u(s))}{\partial t^n}\Psi(s,x(s))ds.$$

Here, we have

$$V^{(0)}(t) = \int_a^t k_1(t,s,u(s))\Phi(s,x(s))ds,$$

$$V^{(1)}(t) = k_1(t,s,u(s))\Phi(t,x(t))\Phi(s,x(s))ds,$$
$$+ \int_a^t \frac{\partial k_1(t,s,u(s))}{\partial t}$$

$$V^{(2)}(t) = 2\Phi(t,x(t))\frac{\partial k_1(t,s,u(s))}{\partial t}$$
$$+ k_1(t,s,u(s))\frac{\partial \Phi(t,x(t))}{\partial t} + k_1(t,s,u(s))x'(t)\frac{\partial \Phi}{\partial x}$$
$$+ \int_a^t \frac{\partial^2 k_1(t,s,u(s))}{\partial t}\Phi(s,x(s))ds,$$

$$V^{(3)}(t) = 3\Phi\frac{\partial^2 k_1(t,s,u(s))}{\partial t} + 3\frac{\partial k_1(t,s,u(s))}{\partial t}\frac{\partial \Phi(t,x(t))}{\partial t}$$
$$+ 2x'(t)\frac{\partial k_1(t,s,u(s))}{\partial t}\frac{\partial \Phi(t,x(t))}{\partial x}$$
$$+ k_1(t,s,u(s))x'(t)\frac{\partial \Phi(t,x(t))}{\partial t}\frac{\partial \Phi(t,x(t))}{\partial x}$$
$$+ k_1(t,s,u(s))(x')^2\frac{\partial^2 \Phi(t,x(t))}{\partial x}$$

$$+k_1(t,s,u(s))\frac{\partial^2\Phi(t,x(t))}{\partial t}$$

$$+\int_a^t \frac{\partial^3 k_1(t,s,u(s))}{\partial t}\Phi(s,x(s))ds,$$

$$\vdots$$

and then

$$V^{(n)}(t)=\sum_{i=0}^{n-1}[h_i(t)\Phi(t,x(t))]^{(n-i-1)} \tag{10}$$

$$+\int_0^t \frac{\partial^n k_1(t,s,u(s))}{\partial t^n}\Phi(s,x(s))ds,$$

where

$$h_i(t)=\frac{\partial^i k_1(t,s,u(s))}{\partial t^i}|_{s=t}\,.$$

Using Liebnitz's rule, we obtain

$$V^{(n)}(t)=\sum_{m=0}^{n-1}\sum_{i=0}^{n-m-1}\binom{n-i-1}{m}h_i^{(n-m-i-1)}(t)[\Phi(t,x(t))]^{(m)} \tag{11}$$

$$+\int_0^t \frac{\partial^n k_1(t,s,u(s))}{\partial t^n}\Phi(s,x(s))ds.$$

Note that, in Eq.(11),

$$\sum_{m=0}^{n-1}\sum_{i=0}^{n-m-1}(\dots)=\sum_{i=0}^{n-i-1}\sum_{m=0}^{n-1}(\dots).$$

If we approximate $x(t),\Phi(t,x(t))$ and $\Psi(t,x(t))$ as in Eqs.(3), (4) and (5), we obtain

$$V^{(n)}(c)=\sum_{i=0}^{n-1}\sum_{i=0}^{n-m-1}\binom{n-i-1}{m}h_i^{(n-m-i-1)}(c)\phi_m \tag{12}$$

$$+\sum_{m=0}^{N}\frac{1}{m!}\phi_m\int_0^c \frac{\partial^n k_1(t,s,u(s))}{\partial t^n}|_{t=c}(s-c)^m ds$$

and

$$F^{(n)}(c)=\sum_{m=0}^{N}\frac{1}{m!}\psi_m\int_0^1 \frac{\partial^n k_2(t,s,u(s))}{\partial t^n}|_{t=c}(s-c)^m ds. \tag{13}$$

Substituting Eqs.(12) and (13) in Eq.(9) gives

$$x^{(n)}(c)=$$

$$y^{(n)}(c)+\lambda_1\sum_{m=0}^{n-1}\sum_{i=0}^{n-m-1}\binom{n-i-1}{m}h_i^{(n-m-i-1)}(c)\phi_m$$

$$+\lambda_1\sum_{m=0}^{N}\frac{1}{m!}\phi_m\int_0^c \frac{\partial^n k_1(t,s,u(s))}{\partial t^n}|_{t=c}(s-c)^m ds \tag{14}$$

$$+\lambda_2\sum_{m=0}^{N}\frac{1}{m!}\psi_m\int_0^1 \frac{\partial^n k_2(t,s,u(s))}{\partial t^n}|_{t=c}(s-c)^m ds.$$

or, briefly,

$$x^{(n)}(c)=y^{(n)}(c)+\lambda_1\sum_{m=0}^{N}H_{nm}\phi_m+\lambda_2\sum_{m=0}^{N}K_{nm}\psi_m \tag{15}$$

where

$$H_{nm}=\begin{cases}\displaystyle\sum_{i=0}^{n-m-1}\binom{n-i-1}{m}h_i^{(n-m-i-1)}(c) \\ \displaystyle +\frac{1}{m!}\int_0^c \frac{\partial^n k_1(t,s,u(s))}{\partial t^n}|_{t=c}(s-c)^m ds & n>m \\ \displaystyle \frac{1}{m!}\int_0^c \frac{\partial^n k_1(t,s,u(s))}{\partial t^n}|_{t=c}(s-c)^m ds & n\le m\end{cases} \tag{16}$$

and

$$K_{nm}=\frac{1}{m!}\int_0^1 \frac{\partial^n k_2(t,s,u(s))}{\partial t^n}|_{t=c}(s-c)^m ds. \tag{17}$$

If we take $n=0,1,\dots,N$, relation (15) reduces to a system of $N+1$ non-linear equations for $N+1$ unknown coefficients $x^{(0)}(c),x^{(1)}(c),\dots,x^{(N)}(c)$, as follows:

$$X=Y+\lambda_1 H\Phi+\lambda_2 K\Psi \tag{18}$$

where

$$X=\begin{bmatrix}x^{(0)}(c)\\x^{(1)}(c)\\x^{(2)}(c)\\\vdots\\x^{(N)}(c)\end{bmatrix},Y=\begin{bmatrix}y^{(0)}(c)\\y^{(1)}(c)\\y^{(2)}(c)\\\vdots\\y^{(N)}(c)\end{bmatrix}\ \Phi=\begin{bmatrix}\phi_0\\\phi_1\\\phi_2\\\vdots\\\phi_N\end{bmatrix},\Psi=\begin{bmatrix}\psi_0\\\psi_1\\\psi_2\\\vdots\\\psi_N\end{bmatrix}$$

And H and K are $(N+1)\times(N+1)$ matrices defined in (16) and (17). The non-linear system of equations obtained in (18) can be solved using standard mathematics toolboxes as MATLAB.

III. The Iterative Method and Its Convergence

Let Q be the subset of the product space $C^\infty([0,1])\times C^\infty([0,1])$ contains all pairs $(x(.),u(.))$, which satisfy Eq.(2). Also, let $Q_{m,n}$ be the subset of Q consisting of all pairs $(x_m(.),u_n(.))$, where $u_n(.)$ is a parameterized control function as the following polynomial

$$u_n(t)=\sum_{i=0}^{n}a_i t^i, \tag{19}$$

and $x_m(.)$ is the extracted solution of the integral equation (2), which is considered as a polynomial of degree at most m

$$x_m(t)=\sum_{j=0}^{m}e_j(a_0,a_1,\dots,a_n)t^j. \tag{20}$$

Here, $e_j:R^n\to R, j=0,1,\dots,m$ are continuous functions. Now, we consider the minimizing of J on $Q_{m,n}$

with $\{a_k\}_{k=0}^{n}$ as unknowns. This is obviously an optimization problem in $n+1$ dimensional space

$$\{(a_0,a_1,...,a_n)\in R^{n+1}:$$

$$a_0=u_n(0)=u_0,\sum_{k=0}^{n}a_k=u_n(1)=u_1\}$$

and $J(x_m,u_n)$ may be considered as a function $J(a_0,a_1,...,a_n)$.

Suppose, $(x_m^*(.),u_n^*(.))$ be the solution of minimizing J on $Q_{m,n},m=1,2,...;\ n=1,2,...$, then the polynomial form of $u_n^*(.),n=1,2,...$ (19) and using (20) allow us to apply the proposed method (in Section 2) for extracting polynomial solution of (20), which results in obtaining a sequence of trajectory functions $\{x_m^*(...)\}_{m=1}$ as Taylor series, and finally to achieve a minimizing sequence $\{(x_m^*(.),u_n^*(.)\}_{m,n}$.

Lemma 1. If $\alpha_{m,n}=\inf_{Q_{m,n}} J$ for $m,n=1,2,...$, then $\{\alpha_{m,n}\}_{m,n=1}^{\infty}$ is a convergent sequence.

Proof. Proof in [16, 19].

Now, it can be concluded that $\{\alpha_{m,n}\}$ is convergent, because it is a nondecreasing and bounded from below sequence.

Theorem 1. If $\lim_{m,n\to\infty}\alpha_{m,n}=\alpha$ then $\alpha=\inf_Q J$.

Proof. Proof in [16, 19].

The above discussion and results can be summarized in a numerical algorithm for obtaining the approximate solutions for the optimal control (1) subject to Eq.(2).

Algorithm 1. Choose $\varepsilon_1>0$ and $\varepsilon_2>0$ for the accuracy of the solution.

- **Step 1.** Let $m,n,k=1,\ u_1(t)=a_0+a_1t$, $x_1(t)=e_0+e_1(t)$ and $\alpha_1=J(x(1),u(1))$, where $e_0=e_0(a_0,a_1)$ and $e_1=e_1(a_0,a_1)$.
- **Step 2.** Let $m=m+1$ and $k=k+1$ and find $\alpha_k=\inf_{Q_{m,n}} J$.
- **Step 3.** If $|\alpha_k-\alpha_{k-1}|<\varepsilon_1$ then go to Step 4, otherwise go to Step 2.
- **Step 4.** Let $n=n+1$ and $k=k+1$, find $\alpha_k=\inf_{Q_{m,n}} J$ and go to Step 5.
- **Step 5.** If $|\alpha_k-\alpha_{k-1}|<\varepsilon_2$ then stop, otherwise go to Step 4.

IV. Numerical Experiments

Now, we show the efficiency of the method described using the following examples. In all examples, the approximate solutions will be compared with the exact solutions.

Example 1. Let us first consider the optimal control problem governed by Fredholm-Hammerstein integral equation as follows

$$Minimize\ \ J=\int_0^1(x(t)-1-t^2)^2+(u(t)-1-t)^2dt, \tag{21}$$

subject to

$$x(t)=y(t)+\int_0^1 t^2\times u(s)\times\Psi(s,x(s))ds. \tag{22}$$

where $y(t)=-\dfrac{31t^2}{30}+t+1$ and $\Psi(t,x(t))=t+x(t)$.

The exact optimal solutions of (21)-(22) are

$$x^*(t)=1+t^2\ \ and\ \ u^*(t)=1+t,$$

with the optimal criterion

$$J=J(x^*(t),u^*(t))=0.$$

In the first iteration, i.e. for $n=1$ and $m=1$, we have

$$X=\begin{bmatrix}x^{(0)}(c)\\x^{(1)}(c)\end{bmatrix},Y=\begin{bmatrix}1\\0\end{bmatrix},K=\begin{bmatrix}0&0\\0&0\end{bmatrix},$$

$$\Psi=\begin{bmatrix}x^{(0)}(c)\\1+x^{(1)}(c)\end{bmatrix}.$$

In the next iteration and for $n=1$ and $m=2$, we get

$$X=\begin{bmatrix}x^{(0)}(c)\\x^{(1)}(c)\\x^{(2)}(c)\end{bmatrix},\quad Y=\begin{bmatrix}1\\0\\-3.8333\end{bmatrix},$$

$$K=\begin{bmatrix}0&0&0\\0&0&0\\2a_0+a_1&a_0+(2a_1)/3&a_0/3+a_1/4\end{bmatrix},$$

$$\Psi=\begin{bmatrix}x^{(0)}(c)\\1+x^{(1)}(c)\\x^{(2)}(c)\end{bmatrix}.$$

So by using the iteration Algorithm 1, the numerical results are illustrated in Table 1 and Figures 1-4.

Table 1: The Approximate-Analytical results for Example 1

$Iter$	n	m	$x(t)$	$u(t)$	J
1	1	1	1	$1+t$	0.2
2	1	2	$1+t^2$	$1+t$	0

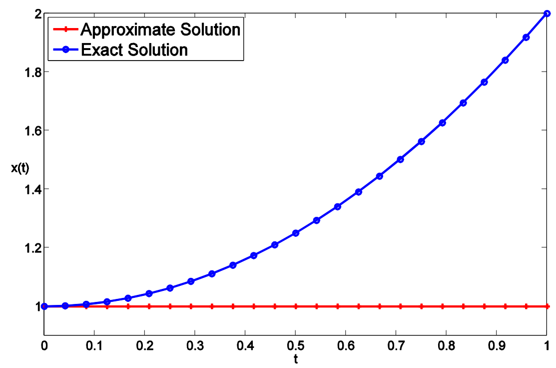

Figure 1: Exact and approximate trajectory functions for Example 1, $n=1, m=1$.

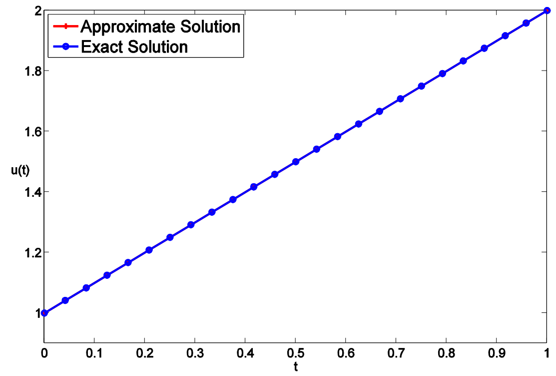

Figure 2: Exact and approximate control functions for Example 1, $n=1, m=1$.

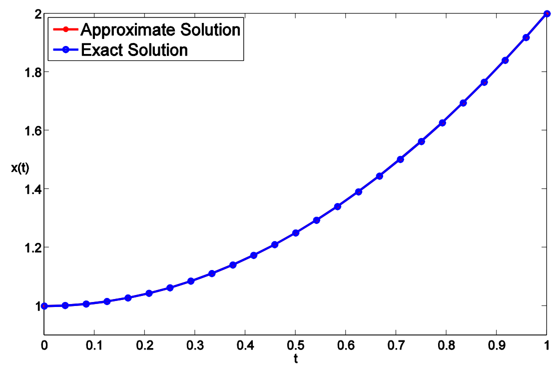

Figure 3: Exact and approximate trajectory functions for Example 1, $n=1, m=2$.

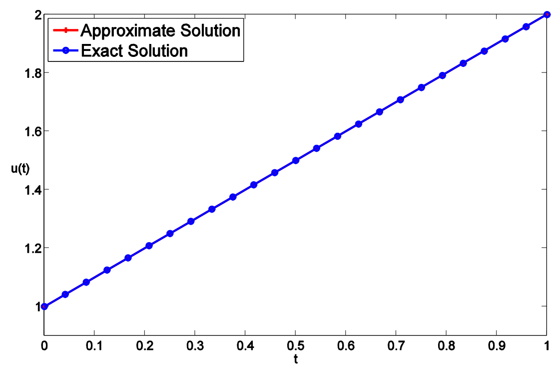

Figure 4: Exact and approximate control functions for Example 1, $n=1, m=2$.

Example 2. We consider the optimal control problem governed by Volterra-Hammerstein integral equation as follows

$$Minimize\ \ J=\int_0^1 (x(t)-t-t^2)^2+(u(t)-1-t^2)^2 dt, \quad (23)$$

subject to

$$x(t)=y(t)+\int_0^t t\times u(s)\times\Phi(s,x(s))ds. \quad (24)$$

where $y(t)=t-t^2\left(\frac{t^4}{5}+\frac{t^3}{2}+\frac{2t^2}{3}+t\right)$ and $\Phi(t,x(t))=1+t+x(t)$.

The exact optimal solutions of the problem (23)-(24) are

$$x^*(t)=t+t^2 \ \ and \ \ u^*(t)=1+t^2,$$

with the optimal criterion

$$J=J(x^*(t),u^*(t))=0.$$

Using the iteration Algorithm 1, the numerical results are illustrated in Table 2 and Figures 5-10.

Table 2: The Approximate-Analytical results for Example 2

Iter	*n*	*m*	$x(t)$	$u(t)$	J
1	1	1	t	$0.8334+t$	0.2056
2	1	2	$t+0.9074t^2$	$0.9074+0.8889t$	0.0086
3	2	2	$t+t^2$	$1+t^2$	0

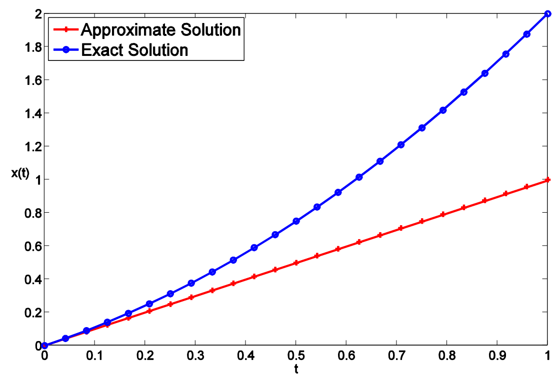

Figure 5: Exact and approximate trajectory functions for Example 2, $n=1, m=1$.

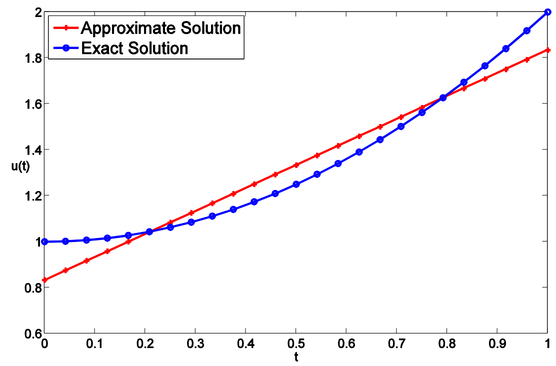

Figure 6: Exact and approximate control functions for Example 2, $n=1, m=1$.

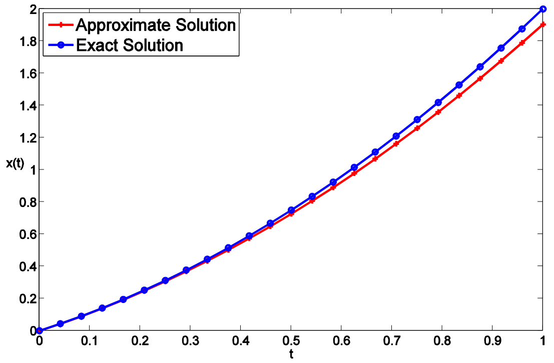

Figure 7: Exact and approximate trajectory functions for Example 2, $n=1, m=2$.

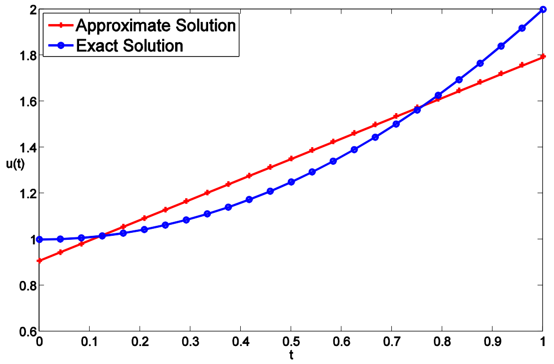

Figure 8: Exact and approximate control functions for Example 2, $n=1, m=2$.

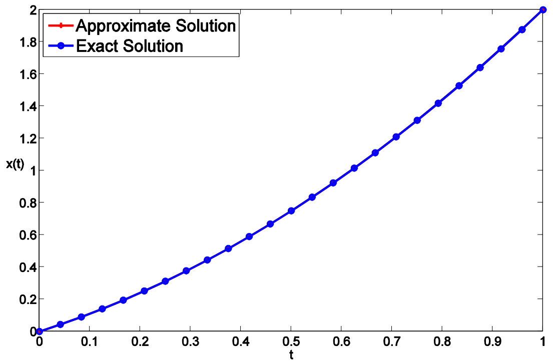

Figure 9: Exact and approximate trajectory functions for Example 2, $n=2, m=2$.

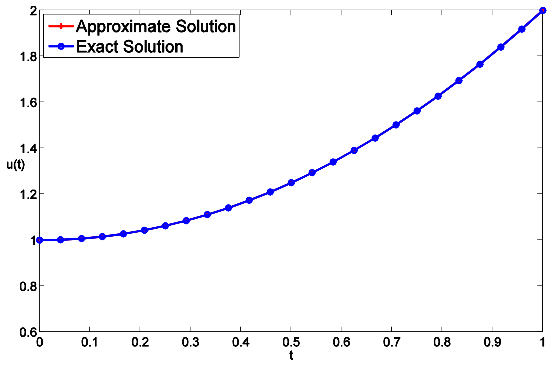

Figure 10: Exact and approximate control functions for Example 2, $n=2, m=2$.

Example 3. Finally, Let us consider the optimal control problem governed by Volterra-Fredholm-Hammerstein integral equation as follows

$$Minimize \;\; J = \int_0^1 (x(t)-t-t^2)^2 + (u(t)-1-t)^2 dt, \quad (25)$$

subject to

$$x(t) = y(t) + \int_0^t exp(t)\times u(s)\times \Phi(s, x(s))ds \quad (26)$$
$$+\int_0^1 exp(t)\times u(s)\times \Psi(s, x(s))ds.$$

where $y(t) = t - \frac{15e^t}{4} + t^2 - \frac{te^t(t^3+4t^2+6t+4)}{4}$,

$\Phi(t, x(t)) = 1 + t + x(t)$ and $\Psi(t, x(t)) = 1 + t + x(t)$.

The exact optimal solutions of (25)-(26) are

$$x^*(t) = t + t^2, \;\; and \;\; u^*(t) = 1 + t,$$

with the optimal criterion

$$J = J(x^*(t), u^*(t)) = 0.$$

Table 3 and Figures 11-14 give the numerical results for example 10.

Table 3: The Approximate-Analytical results for Example 3

$Iter$	n	m	$x(t)$	$u(t)$	J
1	1	1	$0.0256 + 1.6670t$	$1.6004 + 5.9824E-6t$	0.0155
2	1	2	$t+t^2$	$1+t$	0

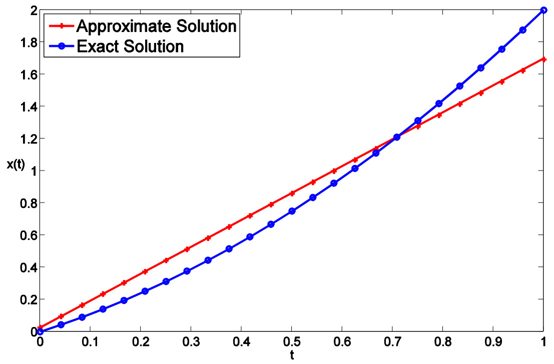

Figure 11: Exact and approximate trajectory functions for Example 3, $n=1, m=1$.

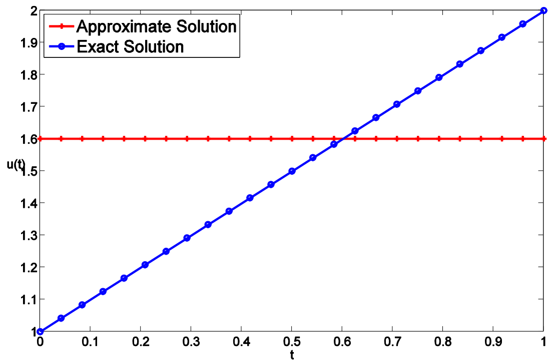

Figure 12: Exact and approximate control functions for Example 3, $n=1, m=1$.

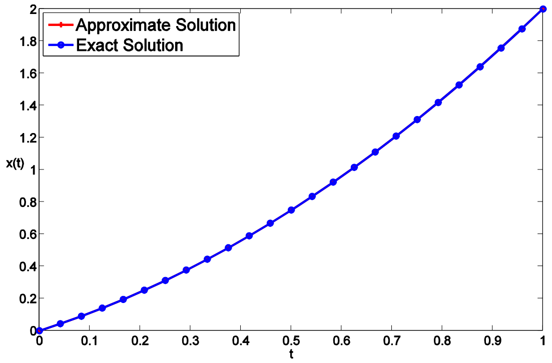

Figure 13: Exact and approximate trajectory functions for Example 3, $n=1, m=2$.

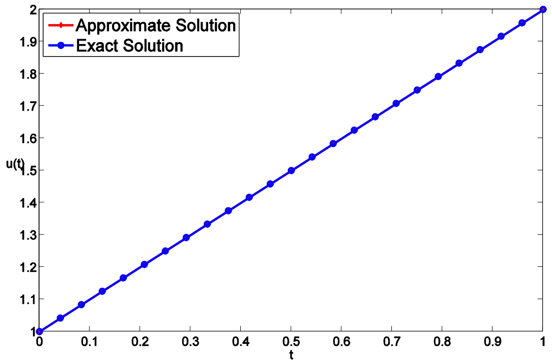

Figure 14: Exact and approximate control functions for Example 3, $n=1, m=2$.

V. Conclusion

Optimal control problems governed by non-linear Hammerstein integral equations are usually difficult to solve analytically and so it is necessary to obtain the approximate solutions. The present method is effective for cases where the known functions have sufficient derivatives in the given interval. One of the advantages of this method is that the optimal solutions, i.e. the trajectory and control functions, are expressed as a Taylor series truncated at $t=c$. Therefore $x(t)$ and $u(t)$ can easily be evaluated for arbitrary values of t with a low computation at effort. This method will not work for cases where the given functions do not have enough derivatives. An interesting feature of this method is that we obtain analytical solution in many cases, as shown in the examples.

References

[1] M.A. Abdou. "On the solution of linear and nonlinear integral equation". *Appl. Math. Comput*., 146, pp. 857-871.
[2] S. Anderyance and M. Hadizadeh, "The Computational Efficiency Of Walsh Approximation For Two-Dimensional Volterra integral equations". *Asian-European Journal of Mathematics*, 4, pp. 263-270.
[3] E. Babolian, J. Biazar and A.R. Vahidi, "The decomposition method applied to systems of Fredholm integral equations of the second kind". *Appl. Math. Comput*., 148, pp. 443-452.
[4] C. Homescu and I.M. Navon, "Optimal control of flow with discountinuos". *J. of computational physics*, 187, pp. 660-682.
[5] M. I. Gil, "Positivity of Green's Functions to Volterra Integral and Higher Order Integro-Differential Equations". *Analysis and Applications*, 7, pp. 405-418.
[6] S.C. Huang and R.P. Shaw, "The Trefftz method as an integral equation". *Adv. Eng. Software*, 24, pp. 57-63.
[7] S. Jiang and V. Rokhlin, "Second kind integral equations for the classical potential theory on open surface II". *J. Comput. Phys*., 195, pp. 1-16.
[8] T. Kailat, "Some integral equations with "Nonrational" kernels". *IEEE, Trans. Inf. Th. IT*., 12, pp. 442-447.
[9] P.K. Kythe and P. Puri, *Computational Methods of Linear Integral Equations*. Birkhauser Bosten, c/o Springer-Verlag, New York (2002).
[10] D. Liang and B. Zhang, "Numerical analysis of graded mesh methods for a class of second kind integral equations on real line". *J. Math. Anal. Appl*., 294, pp. 482-502.
[11] Y. Mahmoudi, "Taylor polynomial solution of non-linear Volterra-Fredholm integral equation". *Int. J. of Computer Mathematics*, 82, pp. 881-887.
[12] K. Maleknejad and M. Karami, "Using the WPG method for solving integral equations of the second kind". *Appl. Math. Comput*., 166, pp. 123-130.
[13] K. Maleknejad and M. Karami, "Numerical solution of non-linear Fredholm integral equations by using multi-wavelets in the Petrov-Galerkin method". *Appl. Math. Comput*., 168, pp. 102-110.
[14] K. Maleknejad and Y. Mahmoudi, "Numerical solution of linear Fredholm integral equations by using hybrid Taylor and Block-Pulse functions". *Appl. Math. Comput*., 149, pp. 799-806.
[15] K. Maleknejad and Y. Mahmoudi, "Taylor polynomial solution of high-order non-linearVolterra-Fredholm integro-differential equations". *Appl. Math. Comput*., 145, pp. 641-653.
[16] O. S. Fard and M. Sanchooli," Taylor solver for Fredholm optimal control problems". *J. of Adv. Research in Differential Equations*, 2, pp. 10-20.
[17] J. Rashidinia and A. Parsa, "Semi-Orthogonal Spline Scalling Functions For Solving Hammerstein Integral Equations". *Int. J. of Wavelets. Multiresolution and Information Processing (IJWMIP)*, 9, pp. 427-443.
[18] J. Saberi-Nadjafi and M. Tamamgar, "Modified Homotopy Perturbation Method for Solving Integral Equations". *Int. J. of Modern Physics B (IJMPB)*, 24, pp. 4741-4746.
[19] M. Sanchooli and O. S. Fard, "Numerical Scheme for Fredholm Integral Equations Optimal Control Problems via Bernstein Polynomials". *Australian J. of Basic and Applied Sciences*. 4, pp. 5675-5682.
[20] I. Y.M. Stems, K.J.E. Versyck and J.F.M. Van Impe, "Optimal control theory: a generic tool for identification and control of (Bio-)chemical reactors". *Annul rewievs of control*, 26, pp. 57-73.
[21] W. Wang, "A new mechanical algorithm for solving the second kind of Fredholm integral equation". *Appl. Math. Comput*., 172, pp. 946-962.
[22] S. Yalcinbas and K. Erdem, "Approximate solutions of Nonlinear Volterra Integral Equation Systems". *Int. J. of Modern Physics B (IJMPB)*, 24, pp. 6235-6258.
[23] S.C. Yang, "An investigation into integral equation methods involving nearly singular kernels for acoustic scattering". *J. Sound Vib*. 234, pp. 225-239.
[24] T. Zdenek, T. Vosegaard, C. Kehlet, N. Khaneja, S. J. Glaser and N. C. Nielsen, "Optimal control in NMR spectroscopy: Numerical implementation in SIMPSON". *Journal of Magnetic Resonance*, 197, pp. 120-134.
[25] S. Zhang, *Integral Equations*. Chongqing press, Chongqing, China 1987.

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