

Free Vibration of Thin Film Cantilever Beam

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Abstract— Measuring of thin film properties is difficult when compared to bulk materials. A straight, horizontal cantilever beam exposed to free vibrations will vibrate at its characteristics or natural frequencies. In this research paper the vibrations of a thin film cantilever beams will be studied and the equations of vibrations will be derived. The frequency equations are solved using Matlab® to show the output frequencies and the mode shapes related to each frequency

Index Terms—cantilever, the mode shapes

I. INTRODUCTION

A cantilever beam is one of the most fundamental structural and machine components used in many different applications for decades. Cantilever beams are generally beams with one end fixed and the other end free. The length has a much larger dimension when compared with the width and depth. In addition, cantilever beams maybe straight or curved, with rectangular or circular cross sections. Figure 1.1 shows a cantilever beam with a rectangular cross section.

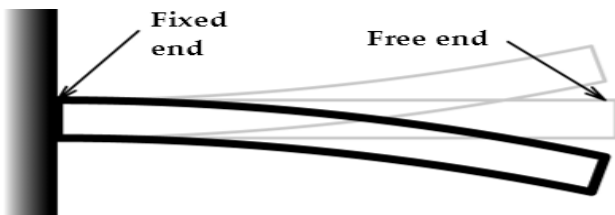


Figure 1.1: Rectangular cantilever beam

Cantilever beam design and shape depends on the application. Its size, material, and weight are different from one application to another. For example, one of the most common applications of a cantilever beam can be shown in figure (1.2) below. The “fixed wing” in meters is designed as a beam for some preliminary analysis to help lift the plane and make it fly [1]



Figure (1.2) Source: Student Online Laboratory through Virtual Experimentation

In Micro Electrical Mechanical Systems (MEMS), micro cantilever beams are used in radio frequency filters and resonator. Beams are exposed to different dynamic loads in

different applications. When a cantilever beam is exposed to a dynamic load, the load will excite the beam to vibrate at its characteristic, or natural, frequencies. Studying vibrations of cantilever beams are very important. It helps to determine the durability concerns (by analyzing dynamic stresses) and noise. This information used to reduce the discomfort and excessive stresses in different applications in which beams are essential components.

Cantilever beams can be more than one layer, in many applications beams are coated with one or more layers. This is possible especially in MEMS applications. In these applications, we may need three layers. One of the MEMS examples is a beam coated with a piezoelectric material. This beam is usually called a “sandwich beam”. In this beam, two piezoelectric layers are used to coat the beam from both sides. As a result, exposing this beam to vibrate will lead to excite the piezoelectric materials to generate a voltage difference between the two piezoelectric layers. This voltage can be used in many applications to measure stress, strain, and get many useful outputs. Another example is a two layer cantilever beam, in which a beam is coated with a different material. For instance, an aluminum cantilever beam is coated with thin film zinc oxide to form a two layers cantilever beam. The coated Aluminum beam has different characteristics and frequency responses than single Aluminum beam. Frequency values will be changed after the beam has been coated. These changes can be used to measure different characteristics. For example, the modulus of elasticity for thin films cannot be measured using conventional methods, because measuring modulus of elasticity for zinc-oxide is very hard and almost impossible. This is due to the thickness of the film and its sensitive characteristics. Thus, using vibration analysis will help in measuring the modulus of elasticity for thin film materials [2].

As seen above, many applications exist for multi-layered beams. In this research paper, we will study the frequency behavior for a one- and a two-layer cantilever beam. The equations of motion, frequency and characteristic equations will be derived. Numerical results will be calculated to see the frequency differences between one and two layered beams. Different materials will be taken to compare the results. In addition, results and future work will be discussed at the end of this paper.

II. LITERATURE REVIEW

In this research our concerns are to calculate and explain the free vibrations of two-layer cantilever beams. However, to make it more obvious we will explain the vibration of a one layer cantilever beam before we move to a two-layer cantilever beam [2].

In this way, we can see the difference in equations and characteristics when we move from one layer to two layer cantilever beams. Our calculations in both cases will depend

on free and un-damped vibration. Thus, the beam isn't exposed to any external force. According to Gorman (1975) "In free vibration a beam undergoes oscillatory motion while free of any external forces, whereas in forced vibration the beam responds to a system of time varying external forces". ([2]-[4]). Free vibration of cantilever beams can happen in an infinite number of mode shapes, each mode has a discrete frequency.

The first frequency which is the lowest one is associated with the first mode; the second frequency is associated with the second mode and so on. However, higher frequencies - third and above - are less significant. This is because they are difficult to excite and the number of points on the beam having zero displacement increase directly with the mode number. To get appreciable amplitude for the higher modes, much more energy is required in this case [5]. We will take a bulk material coated (covered) with thin film in our analysis even though the analysis will be valid for any kind of beams. A thin film is a layer of material ranging from fractions of a nanometer (monolayer) to several micrometers in thickness, and usually the act of applying a thin film to a surface is called thin-film deposition. For example, an Aluminum cantilever beam can be coated with Zirconate Titanate (PZT) thin film to form a two layers cantilever beam. Dealing with thin film materials needs more attention because thin films characteristics are changed rapidly, with non-linearity, and sensitivity. One important thing about dealing with double layer cantilever beam that the thin film beam stiffness should be different from the main beam (bulk material), so the frequency shift can be noticed ([5]-[7]).

Two main methods are used to obtain the solutions for a free vibration cantilever beam. The first one depends on solving differential equations for the equilibrium between inertia forces and elastic restoring forces subject to boundary conditions. The second method called the energy method, this method depends on the fact that the sum of the potential energy and the kinetic energy is always constant. [2].

In the next chapter we will use the differential equation method to derive the equations for free vibration of a cantilever beam. These equations will be the main reference for the next chapters in which it will be used to derive the equations for two layers beam, and obtain the frequencies and mode shapes.

III. CANTILEVER BEAM FREE VIBRATION THEORY

A. Equation of Motion

We assume a cantilever beam with length at least 20 times the average depth, and the beam vibrates transversely in the z-direction as shown in figure 2.1 below. It makes the problem mathematically one dimensional with no torsional vibrations. We take a small differential beam element of length dl from the beam as shown in figure 2.2. By developing the free body diagram for this element, we can see that there is a shear force and a bending moment in both sides. These quantities vary along the beam with time (t). In addition, in free vibration case there is no force, $p=zero$ ([1]-[3]).

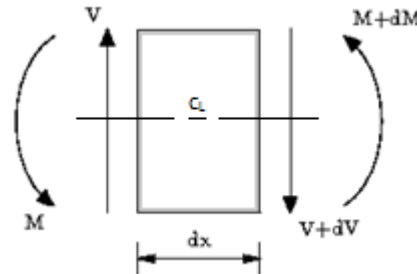
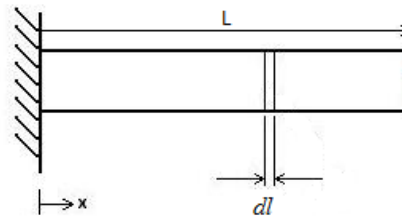


Figure 2.1: Cantilever beam Figure 2.2: Differential beam element

The summation of forces around the center line is:

$$\sum Fy = m \frac{\partial^2 w}{\partial t^2}$$

$$\boxed{V - (V + \frac{\partial V}{\partial x} dx) = \rho A dx \cdot \frac{\partial^2 w}{\partial t^2}} \rightarrow (1)$$

Where ρ is the mass per unit volume and A is the cross sectional area, the beam is assumed to be under small vibrating amplitude; thus, the slope of the centerline $\frac{\partial w}{\partial x}$ is

everywhere small and $\frac{\partial}{\partial l} = \frac{\partial}{\partial x}$ as well. Cancelling terms

and dividing by dx (2.1) becomes:

$$\boxed{-\frac{\partial V}{\partial l} = \rho A \frac{\partial^2 w}{\partial t^2}} \rightarrow (2)$$

Take the moments about the center:

$$\sum M = 0$$

$$\boxed{-M + (M + \frac{\partial M}{\partial x} dx) - Vdx = 0} \rightarrow (3)$$

Cancelling terms and divided by dx , equation (2.3) becomes:

$$\boxed{V = \frac{\partial M}{\partial l}} \rightarrow (4)$$

Back to Euler-Bernoulli equation for linear elastic materials:

$$\boxed{M = EI \frac{\partial^2 w}{\partial l^2}} \rightarrow (5)$$

Where $\frac{\partial^2 w}{\partial l^2}$ is the small slope curvature, E is the Young's modulus of elasticity, and I the second moment of inertia area. Substituting (5) into (4) gives:

$$\boxed{V = \frac{\partial}{\partial l} (EI \frac{\partial^2 w}{\partial t^2})} \rightarrow (6)$$

Now substituting (6) into (2) gives:

$$\frac{\partial^2}{\partial l^2} (EI \frac{\partial^2 w}{\partial l^2}) + \rho A \frac{\partial^2 w}{\partial t^2} = 0$$

$$EI \frac{\partial^4 w}{\partial l^4} + \rho A \frac{\partial^2 w}{\partial t^2} = 0 \rightarrow (7)$$

This is the equation of motion for homogeneous, constant cross section, and free vibration cantilever beam. Next, we will solve this equation for free vibration cantilever beam.

IV. SOLUTION OF THE DIFFERENTIAL EQUATION

Using the method of separation of variables we assume a solution of the form:

$$w(l, t) = X(l) \cdot Q(t) \rightarrow (8)$$

X: independent on time, Q: independent of position

We rearrange equation (7) into two parts, one depends on position and the other depends on time:

$$EI \frac{\partial^4 (X(l) \cdot Q(t))}{\partial l^4} = -m \frac{\partial^2 (X(l) \cdot Q(t))}{\partial t^2}$$

$$X(l) = C_1 \sin(\beta l) + C_2 \cos(\beta l) + C_3 \sinh(\beta l) + C_4 \cosh(\beta l) \rightarrow (11)$$

$$Q(t) = A_1 \sin(\omega t) + A_2 \cos(\omega t) \rightarrow (12)$$

Because we are interested in finding βl , we will calculate the constants $C_1, C_2, C_3,$ and C_4 for equation (11) only. To find these constants and solve the equation, boundary conditions and initial conditions should be determined. However, the above equations are valid for any beam with arbitrary boundary conditions. In our case we will solve these equations using the boundary conditions for a cantilever beam [8]

V. BOUNDARY CONDITIONS FOR A CANTILEVER BEAM IN FREE VIBRATION

In this section we will derive the solution for the cantilever beam shown in figure 2.3 below. This beam has a length (L), a thickness (t), and a width (w).

The solution here is for the one layer cantilever beam. We will use this solution to derive the solution for two-layer cantilever beams. Applying the following boundary conditions for this beam:

a- At the fixed end the deflection (z) and the slope are zeros.

$$w(0, t) = 0 \rightarrow \frac{\partial w}{\partial l}(0, t) = 0 \rightarrow (13)$$

b- At the free end the moment(M) and the shear force(V) are zeros

$$\frac{\partial^2 w}{\partial l^2}(L, t) = 0 \rightarrow \frac{\partial^3 w}{\partial l^3}(L, t) = 0 \rightarrow (14)$$

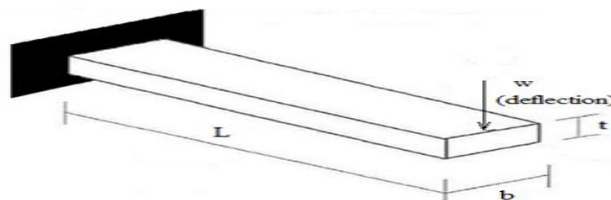


Figure (2.3): One layer cantilever beam

Implement equations (13) and (14) into equation (8), we get X (l):

$$\begin{aligned} X(0, t) = 0 &\rightarrow \frac{dX}{dl}(0, t) = 0 \\ \frac{d^2 X}{dl^2}(L, t) = 0 &\rightarrow \frac{d^3 X}{dl^3}(L, t) = 0 \end{aligned} \rightarrow (15)$$

Divide two sides by $m \cdot X(l) \cdot Q(t)$ we get:

$$\frac{EI}{mX(l)} \cdot \frac{\partial^4 X(l)}{\partial l^4} = - \frac{1}{Q(t)} \cdot \frac{\partial^2 Q(t)}{\partial t^2} \rightarrow (9)$$

Each side is constant and equal to ω^2 which is the natural frequency of the cantilever beam with real quantity, take the left side of the equation and rearrange it to become:

$$\frac{\partial^4 X(l)}{\partial l^4} - \frac{m\omega^2}{EI} X(l) = 0$$

$$\beta^4 = \frac{m\omega^2}{EI} \rightarrow (10)$$

$\beta^4 \equiv$ The corresponding non-dimensional frequency for infinite number of values. To determine β we will find the general solution for the differential equation (8).

The general solution of equation (2.8) is given by:

We will use the boundary conditions (15) to solve equation (11) by differentiating it three times.

$$\frac{dX}{dl} = \beta(C_1 \cos(\beta l) - C_2 \sin(\beta l) + C_3 \cosh(\beta l) + C_4 \sinh(\beta l)) \rightarrow (16)$$

$$\frac{d^2 X}{dl^2} = \beta^2(-C_1 \sin(\beta l) - C_2 \cos(\beta l) + C_3 \sinh(\beta l) + C_4 \cosh(\beta l)) \rightarrow (17)$$

$$\frac{d^3 X}{dl^3} = \beta^3(-C_1 \cos(\beta l) + C_2 \sin(\beta l) + C_3 \cosh(\beta l) + C_4 \sinh(\beta l)) \rightarrow (18)$$

Substitute the boundary conditions in equations (11) and (16) we get :

$$X(0, t) = 0 = C_1 \sin(0) + C_2 \cos(0) + C_3 \sinh(0) + C_4 \cosh(0)$$

$$C_2 = -C_4 \rightarrow (19)$$

$$\frac{dX}{dl} \downarrow_{l=0} = 0 = \beta(C_1 \cos(0) - C_2 \sin(0) + C_3 \cosh(0) + C_4 \sinh(0))$$

$$C_1 = -C_3 \rightarrow (20)$$

Now substitute equations (19) and (20) with boundary conditions into equations (17) and (18) to get:

$$\frac{d^2 X}{dl^2} \downarrow_{l=L} = 0 = -C_1 \sin(\beta L) - C_2 \cos(\beta L) - C_1 \sinh(\beta L) - C_2 \cosh(\beta L) \rightarrow (21)$$

$$\frac{d^3 X}{dl^3} \downarrow_{l=L} = 0 = -C_1 \cos(\beta L) + C_2 \sin(\beta L) - C_1 \cosh(\beta L) - C_2 \sinh(\beta L) \rightarrow (22)$$

These equations can be written in a matrix form:

$$\begin{pmatrix} \sin(\beta L) + \sinh(\beta L) & \cos(\beta L) + \cosh(\beta L) \\ \cos(\beta L) + \cosh(\beta L) & -\sin(\beta L) + \sinh(\beta L) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

To solve this matrix we will take the non-trivial solution at the resonant frequency in which the determinant of the matrix must equal to zero; thus:

$$\sinh^2(\beta L) - (\sin^2(\beta L) + \cosh^2(\beta L)) - 2 \cos(\beta L) \cosh(\beta L) - \cosh^2(\beta L) = 0$$

$$1 + \cos(\beta L) \cosh(\beta L) = 0 \rightarrow (23)$$

$\beta > 0$, $\beta \neq 0$ (No vibration when the beam at rest) $\beta \neq \text{negative}$ (Negative values lead to redundancy)

This is a non-linear equation with infinite number of roots (i.e. continuous system). Solving this non-linear equation (** See Appendix-A) we get:

$$\beta L = 1.875, 4.6904, 7.548, \dots$$

Back to equation (2.10) and rearrange it for ω :

$$\omega = \frac{(\beta L)^2}{L^2} \sqrt{\frac{EI}{\rho A}} \rightarrow (24)$$

Where: E \equiv Young modulus of elasticity (Pa)

$$I = \frac{1}{12} bt^3 \equiv \text{Moment of inertia-Area for a rectangular cross-section (m}^4 \text{)}$$

$$\rho \equiv \text{Mass density (Kg / m}^3 \text{)}$$

$$A = t.b \equiv \text{Area of the cross section (m}^2 \text{)}$$

$$L, b, t \equiv \text{Length, width, and thickness respectively}$$

This equation computes the frequencies for a one layer cantilever beam. Infinite number of frequencies can be calculated using this equation. In the next section we will derive the equation for a two-layer cantilever beam [10].

VI. VIBRATION OF A TWO LAYER CANTILEVER BEAM

We discussed the vibration for one layer cantilever beam, but many recent applications use more than one layer in which the beam is coated to become a composite material.

We will get a new material that is different from the old material. This is very helpful in many applications like automotive engineering, MEMS and NEMS, and biomedical equipment. The use of composite materials can reduce weight, increase efficiency and improve output response. As

a result, instead of using Steel for a cantilever beam, we can use two other materials (composite) and get the desired output. In our case get the same frequency response.

Thin film materials usually in micro-meters or even nano-meters are using to coat other materials, in this case we will take a beam coated with a thin film to form a new two layer beam as shown in figure (2.4) below [9]. The new beam will have new characteristics and new specifications. This two layer beam has two materials. Each material has different density, and Young modulus of elasticity. In addition, inertia and thickness differ from each layer to the other. Both layers still have the same length and width, and both of them will vibrate at the same frequency.

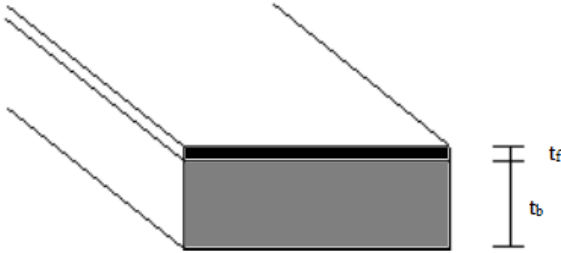


Figure (2.4): Thin film cantilever beam

Both materials will vibrate at the same frequency, so the frequency will be shifted to a new value (i.e. modified). This frequency shift will be very small, because the thin film is in micro-scale. We will take the case of a two layer beam and derive the necessary equations. More layers can be added easily and implemented to the equations. Lets take the beam in figure (2.4), where (f) stands for the thin film and (b) stands for the coated beam (bulk), the beam has:

$$\rho_b \equiv \text{Density of the bulk material in } (Kg / m^3)$$

$$\rho_f \equiv \text{Density of the thin film material in } (Kg / m^3)$$

$$E_b \equiv \text{Young Modulus of Elasticity for the bulk material } (Pa)$$

$$E_f \equiv \text{Young Modulus of Elasticity for the thin film } (Pa)$$

$$t_f, t_b \equiv \text{Thin film thickness and Bulk material thickness respectively (m)}$$

Back to frequency equation (2.24), the modulus, inertia, and density will become:

$$I_1 = \sum_{k=1}^N b \rho^k (h_k - h_{k-1}) \rightarrow (25) \text{ (Qatu \& Leissa, 2011, p375)}$$

$$D_{11} = \sum_{k=1}^N b \bar{Q}_{11}^k \left(\frac{h_k^3 - h_{k-1}^3}{3} \right) \rightarrow (26) \text{ (Qatu \& Leissa, 2011, p375)}$$

$$\bar{Q}_{11}^k = E_{11} \left(\frac{1 - \nu_{23} \nu_{32}}{\Delta} \right) \text{ (Qatu \& Leissa, 2011, p367)}$$

After we define the equations for the frequency for both cases (one layer and two layer beams), we need to know the mode shapes of the frequencies.

$$\Delta = 1 - \nu_{12} \nu_{21} - \nu_{23} \nu_{32} - \nu_{31} \nu_{13} - 2 \nu_{21} \nu_{32} \nu_{13} ;$$

$$\nu_{12}, \nu_{21}, \nu_{23}, \nu_{32}, \nu_{31}, \nu_{13} \equiv \text{The poisson's ratios}$$

Where I_{11} is the modified mass per unit length, \bar{Q}_{11}^k is the material Elasticity, and N is the number of layers. These are the general equations for any number of layers, in our case for a two-layer beam, we will substitute N=2, so equations (25) & (26) become:

$$I_1 = \sum_{k=1}^2 b \rho^k (t_k - t_{k-1})$$

$$I_1 = b \left[(\rho^b)(t_1 - t_0) + \rho^f (t_2 - t_1) \right] \rightarrow (27)$$

Where t_0, t_1, t_2 are shown in figure (2.5)

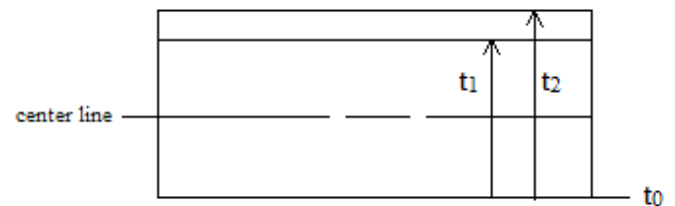


Figure (2.5): Two layers cantilever beam thickness

The new modulus of elasticity will be:

$$D_{11} = \sum_{k=1}^2 b \bar{Q}_{11}^k \left(\frac{t_k^3 - t_{k-1}^3}{3} \right)$$

$$D_{11} = b \left[\bar{Q}_{11}^1 \left(\frac{t_1^3 - t_0^3}{3} \right) + \bar{Q}_{11}^2 \left(\frac{t_2^3 - t_1^3}{3} \right) \right]$$

Because we are dealing with one direction only and for more simplicity, we consider $\Delta \approx 1$ and $\nu_{23}, \nu_{32} = 0$. Thus, the equation becomes:

$$D_{11} = b \left[E_{11}^1 \left(\frac{t_1^3 - t_0^3}{3} \right) + E_{11}^2 \left(\frac{t_2^3 - t_1^3}{3} \right) \right] \rightarrow (28)$$

Where E is the modulus of elasticity of each material, t is the layer thickness, and b is the beam width.

We can implement these equations now in the frequency equation (2.24) we obtained earlier

$$\omega = \frac{(\beta L)^2}{L^2} \sqrt{\frac{D_{11}}{I_1}} \rightarrow (28)$$

This is the frequency response for a two layer cantilever beam. We can add more layers by following the previous steps for two layers.

VII. PROPERTIES OF THE MODE SHAPES

There is infinite number of non-dimensional frequencies and eigen values for each problem. Back to equations (2.11), we can rearrange it for the eigen vectors to determine the corresponding eigen functions. (Leissa & Qatu, 2011, p.110), the equation will become:

$$X(l) = C_1 \sin(\beta l) + C_2 \cos(\beta l) + C_3 \sinh(\beta l) + C_4 \cosh(\beta l) \rightarrow (29)$$

Using equations (2.19), (2.20), (2.21) and (2.22) for a cantilever beam and combined them we get:

$$X(l) = \left[\frac{\cosh(\beta l) - \cos(\beta l)}{\sinh(\beta l) - \sin(\beta l)} \cdot [\sinh(\beta l) - \sin(\beta l)] + \cosh(\beta l) - \cos(\beta l) \right] \rightarrow (30) \text{ Where}$$

$$\phi = -\frac{\cosh(\beta l) - \cos(\beta l)}{\sinh(\beta l) - \sin(\beta l)} = 0.734, 1.018, 0.999, \dots$$

So equation (30) becomes:

$$X(l) = [\cosh(\beta l) - \cos(\beta l) - \phi \cdot [\sinh(\beta l) - \sin(\beta l)]] \rightarrow (31)$$

$X(l) \equiv$ the characteristic function for each frequency

Back to equation of motion (8):

$$w(l, t) = X(l) \cdot Q(t) \rightarrow (8)$$

This equation satisfies the boundary conditions in (13) and (14), which is the solution for each frequency; however, the general solution of the equation of motion is the sum of the characteristic vibrations (Volterra, 1965, p.3.18), thus (8) becomes :

$$w_n(l, t) = \sum_{n=1}^{\infty} X_n(l) [A_n \cos(w_n t) + B_n \sin(w_n t)]$$

$$w_n(l, t) = \sum_{n=1}^{\infty} \left\{ \frac{\cosh(\beta l) - \cos(\beta l)}{\sinh(\beta l) - \sin(\beta l)} \cdot [\sinh(\beta l) - \sin(\beta l)] + \cosh(\beta l) - \cos(\beta l) \right\} \cdot [A_n \cos(w_n t) + B_n \sin(w_n t)] \quad * \text{The above equation (32)}$$

is the general solution for any number of modes (n)

To determine the coefficients A_n and B_n , we will use the initial conditions: $w(0, t) = 0, \frac{\partial w}{\partial l}(0, t) = 0$

Both parameters are constants and depend on these conditions.

$$\text{Thus, } \left. \begin{aligned} w(l, t) &= \varphi \\ \frac{\partial w}{\partial l}(l, t) &= \zeta \end{aligned} \right\} \rightarrow \text{where } 0 \leq l \leq L$$

It follows from equation (32) above:

$$\left. \begin{aligned} \sum_{n=1}^{\infty} A_n X_n(l) &= \varphi(l) \\ \sum_{n=1}^{\infty} w_n B_n X_n(l) &= \zeta(l) \end{aligned} \right\} \rightarrow (33)$$

These equations are analogous to the Fourier series expansions. Thus, both equations are determining the series expansions of the functions $\varphi(l), \zeta(l)$ in terms of the characteristic functions X_n (Volterra & Zachmanoglou, 1965, p.318).

Finally, to find the constants A_n and B_n we make use of the orthogonality for the characteristic functions; thus, by integrating these equations with respect to (l) from $0 \rightarrow L$, and multiplying by $X_n(l)$ we get the formulas for A_n and B_n :

$$A_n = \frac{2}{L} \int_0^l \varphi(l) X_n(l) dl$$

$$B_n = \frac{2}{Lw_n} \int_0^L \zeta(l) X_n(l) dl$$

→ (34) (Volterra & Zachmanoglou, 1965, p.913)

In the last equation we can eliminate B_n , because it depends on the initial velocity, and in this case the beam starts at rest.

Thus, $B_n=0$. So for our calculations we will evaluate A_n , because A_n depends on the initial position at $t=0$ only.

In the next chapter we will evaluate the displacement caused by each mode, and the total initial displacement for different cantilever beams, as long as we will draw the mode shapes for each case using MATLAB program.

VIII. EXAMPLES AND RESULTS

MATERIAL PROPERTIES AND SELECTION

We will introduce first some common materials properties. Determining the suitable material is very important in any cantilever beam design, because the material properties play a significant role in the output values and the whole beam vibration. Thin films are sensitive to circumstances and different external disturbances. Their properties are changed rapidly with instability. For instance, the density for a thin film material can be easily affected by temperature, and that results in changing the outputs and the related frequencies as well ([10]-[12]).

In our calculations of beam vibration, we need to determine three main characteristics: beam dimensions (length, width, and thickness), Young Modulus of Elasticity, and material density. All of them are dependent on the application in which the beam will be used. For instance, in our experiment we will use a thin film cantilever beam, a beam coated with a thin layer; the dimensions will vary from micros to centimeters, also the density and the Elasticity should be chosen carefully.

In the next table (3.1) we summarize some of the materials specifications. We concentrate on the density and the elasticity which is important for our calculations.

Table (3.1): Materials properties

Material	Density (kg/m ³)	Young Modulus (GPa)	Reference
Aluminum (AL)	2700	70	CRC Materials Science and Engineering Handbook, p.46
Silicon (Si)	2330	190	J.mater.Res, Vol.12, No.1, Jan1997, p.59
Copper (Cu)	8960	128	Mechanics of Materials 23(1996), p.314

LEAD ZIRCONATE TITANATE (PZT)	7750	10-40	IEEE Micro Electro Mechanical Systems Workshop, Jan-Feb 1991, Nara, Japan, p.118
STAINLESS STEEL(BULK)	7900	200	Proceedings of IEEE, Vol 70, No.5, May 1982, p.421
STAINLESS STEEL (THIN FILM)	7900	240	Thin solid films 290-291(1996), p.363
Platinum	21440	170	IEEE, Micro Electro Mechanical Systems Workshop, Feb 1990, Napa Vally, California, p.174
SILICON NITRIDE (Si ₃ N ₄), THIN FILM	2843-2887	85-105	Sensors and Actuators A, 35(1992), p.153-159
Titanium (Ti) (Bulk)	4510	115	CRC Materials Science and Engineering Handbook, p.47
Titanium (Ti) (Thin film)	4510	110	Thin Solid Films, 270(1995), p.263

It is very important to notice the difference in properties for the same materials. This table shows the common materials used in laboratories and experiments. For the best results we will choose the material with stable elasticity and density. These characteristics in the table are based on experiments, computer analysis, and other technologies; thus, there are differences in the same material. Also, as we note that materials may have two categories, one for bulk and the other for thin film. For instance, Stainless steel has a Young Modulus of 200 GPa for the bulk material, and 240 GPa for the thin film.

IX. CALCULATIONS AND RESULTS

In this section we will calculate the frequencies and mode shapes for one and two layers beam with different materials selection. We will use MATLAB to implement the equations, calculate the frequencies, and plot the mode shapes [12].

The program will calculate the frequencies in Hertz, and can plot up to six mode shapes. The first modes have larger values and the higher vibration modes are not shown.

• **Aluminum cantilever beam (one layer)**

For Aluminum we consider the following characteristics:

$$L=0.12\text{ m } b=0.03\text{ m } t=0.002\text{ m}$$

$$E=70\text{ GPa } \rho=2700\text{ Kg / m}^3$$

1. Natural Frequencies (Hz)

- Mode shape (1) (Hz): 113.420
- Mode shape (2) (Hz): 710.792
- Mode shape (3) (Hz): 1990.239
- Mode shape (4) (Hz): 3900.074
- Mode shape (5) (Hz): 6447.102
- Mode shape (6) (Hz): 9630.854

2. Mode shapes

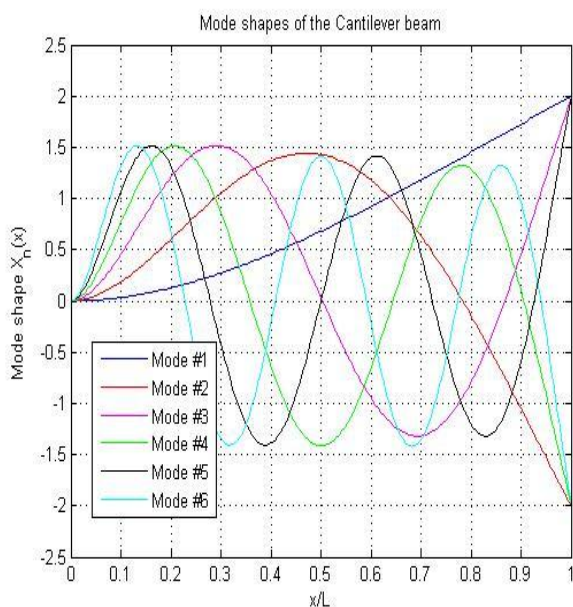


Figure (3.1): Mode shapes for Aluminum beam

Figure (3.1) below shows the displacement caused by each mode at $t=0$ as long as the total displacement for the beam. From these displacements it is obvious that modes higher than the third mode have a little effect on the total displacement.

• **Steel cantilever beam (one layer)**

For Steel we consider the following characteristics:

$$L=0.12\text{ m } b=0.003\text{ m } t=0.002\text{ m}$$

$$E=210\text{ GPa } \rho=7850\text{ Kg / m}^3$$

1. Natural Frequencies (Hz)

- Mode shape (1) (Hz): 116.044
- Mode shape (2) (Hz): 727.235
- Mode shape (3) (Hz): 2036.279
- Mode shape (4) (Hz): 3990.295
- Mode shape (5) (Hz): 6596.244
- Mode shape (6) (Hz): 9853.646

The frequencies depend on the Young Modulus and the density for the material.

2. Mode shapes

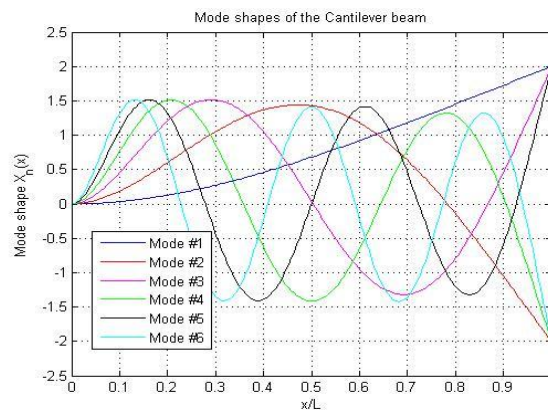


Figure (3.2): Steel beam mode shapes

• **Aluminum coated with Zinc oxide thin film (two layers beam)**

For this beam we consider the following characteristics:

$$L_b = L_f = 0.12\text{ m } b_b = b_f = 0.03\text{ m}$$

$$t_b = 0.002\text{ m } t_f = 0.00002\text{ m}$$

$$E_b = 70\text{ GPa } E_f = 111\text{ GPa}$$

$$\rho_b = 2700\text{ Kg / m}^3 \quad \rho_f = 5680\text{ Kg / m}^3$$

1. Natural Frequencies (Hz)

- Mode shape (1) (Hz): 114.933
- Mode shape (2) (Hz): 720.272
- Mode shape (3) (Hz): 2016.782
- Mode shape (4) (Hz): 3952.088
- Mode shape (5) (Hz): 6533.085
- Mode shape (6) (Hz): 9759.297

Notice the slight difference after the Aluminum coated, because the thickness of the thin film is only 20 micrometers.

2. Mode shapes

Figure (3.3) below shows the mode shapes for a coated beam (two layers). It is so difficult to notice the changes. In addition the mode shapes amplitude is the same, in our work we don't care about the amplitude, because we don't have any damping system to change it; thus, it will be the same in all cases.

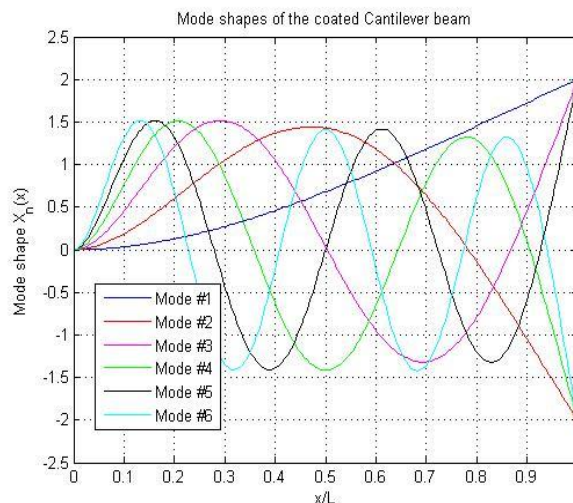


Figure (3.3): Aluminum coated with Zinc oxide mode shapes

• **Steel coated with Silicon nitride (two layers beam)**

For this beam we consider the following characteristics:

$$L_b = L_f = 0.12\text{m} \quad b_b = b_f = 0.03\text{m}$$

$$t_b = 0.002\text{m} \quad t_f = 0.00002\text{m}$$

$$E_b = 210\text{GPa} \quad E_f = 190\text{GPa}$$

$$\rho_b = 7850 \text{ Kg} / \text{m}^3 \quad \rho_f = 2330 \text{ Kg} / \text{m}^3$$

1. Natural frequencies (Hz)

Mode shape (1) (Hz): 117.451

Mode shape (2) (Hz): 736.054

Mode shape (3) (Hz): 2060.973

Mode shape (4) (Hz): 4038.684

Mode shape (5) (Hz): 6676.234

Mode shape (6) (Hz): 9973.138

2. Mode shapes

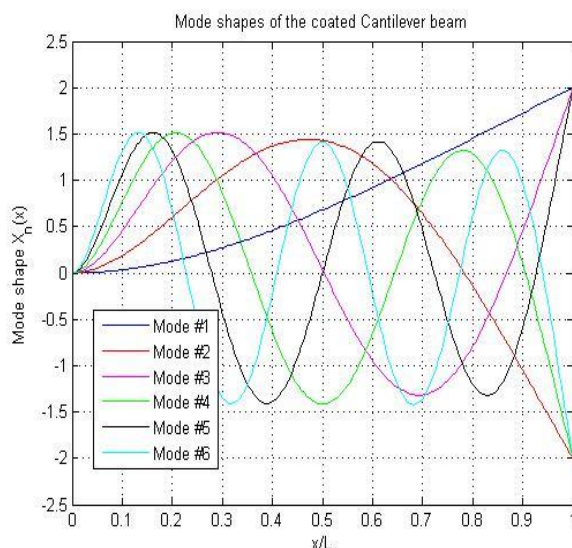


Figure (3.4): Steel coated with Silicon nitride mode shapes

The last examples show the behavior for different cantilever beams according to the frequencies and mode shapes; the MATLAB which is shown in (*Appendix-B) computes the frequencies and mode shapes for any kind of cantilever beam from bulk materials to thin films.

X. SUMMARY AND FUTURE WORK

This research is concerned with the frequencies and mode shapes of a thin film coated cantilever beam exposed to free vibrations. Equations of frequency and mode shapes are derived and solved for different cantilever beams with different materials. For instance, it explained different cases for one and two layer beam, with different materials for each case. We used Mat-Lab® to find the frequencies and the mode shapes related to them, and compared the results for each case. This program is valid for any kind of beams with one or two layers and any kind of materials, the user select the program for one or two layer beam, and then enter the beam dimensions and properties, in a second the program calculate the frequencies and the attached mode shapes for each frequency.

Studying the vibration for composite materials (i.e. two and more different materials) is very important for a lot of

industries including manufacturing companies, and automotive. The cantilever beam is the main component used in many applications and industrial products. Automotive companies are always search for lightweight materials in their cars; lightweight materials and thin films are highly recommended for automotive industries because:

1. It has less exhaust and it prevents high pollution
2. Cost reduction
3. Recycling and environmental friendly
4. Maximize fuel efficiency
5. Reducing vehicle weight

Automotive industries are searching for lighter, stronger, safer, and more environmental friendly materials than current ones. However, studying the frequency responses for the materials is very important for automotive industries as long as the material itself. The automotive companies are concerned with reducing vibrations on the vehicle. Vibrations affect the vehicle durability, operation, and stability. Thus, knowing the vibration will help to make improvements and increase the vehicle durability. Beams generally and cantilever beams specially are one of the main components of the vehicles; thus, studying and analyzing the vibrations of these components are very important in the long term as long as the short term.

A future work will be on measuring the frequencies for real beams in the laboratory and comparing the theoretical results with the practical ones. In addition, it is possible to find material properties that it's hard to find in the basic ways, and see how the frequency changes with external disturbances and effects.

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APPENDICES

• APPENDIX-A: SOLVING NON-LINEAR EQUATION

Solve the non linear equation $1 + \cos(\beta L) \cosh(\beta L) = 0$

Let $x = \beta l$

$$\cos(x) \cosh(x) = -1$$

From the identity

$$\cosh(x + iy) = \cosh(x) \cos(y) + i \sinh(x) \sin(y)$$

$$-1 + i0 = \cosh(x) \cos(y) + i \sinh(x) \sin(y)$$

$$-1 = \cosh(x(1+i))$$

Let $z = x(1+i)$ $\cosh(z) = -1$

From the identity $\cosh(z) = \frac{e^z + e^{-z}}{2}$

$$e^z + e^{-z} = -2$$

$$(e^z)^2 + 1 = -2e^z$$

$$(e^z)^2 + 2e^z + 1 = 0 \rightarrow (e^z + 1)^2 = 0 \rightarrow e^z = -1$$

Use the identity

$$e^{i\pi} = \cos(\pi) + i \sin(\pi) \rightarrow e^z = e^{i\pi} \rightarrow z = i\pi + 2n\pi$$

So, $x(1+i) = \pi i + 2n\pi \rightarrow x = \frac{\pi i + 2n\pi}{1+i}$

Take the conjugate $\rightarrow x = \frac{\pi + 2n\pi}{2} + i \frac{\pi - 2n\pi}{2}$

Take the real part $\rightarrow x = \frac{\pi + 2n\pi}{2}$, for $n=0, 1, 2, 3,$

4.....

• APPENDIX-B :MAT LAB CODES

I. One Layer Cantilever Beam

```
function Cbeam(~)
% Cbeam.m Cantilever Beam calculations
% HELP: This script computes mode shapes and
corresponding natural
```

```
%
clearall;
clc;
closeall;
display
('*****')
display
('*****')
display ('Mohammad Abu-Shams')
display ('School of Engineering and
Technology')
display
('*****')
display
('*****')
display ('This program is ONLY for cantilever
beams with a uniform rectangular cross
section.')
display ('This program computes mode shapes and
crossponding natural')
display ('frequencies up to 6 modes for a
rectangular cantilever beam.')
display ('The cross section should be
rectangular, enter 1 if this is correct, other
wise the program will be closed')
display ('To see a quick example , enter 2')
CS=input(' Enter your choice: ');
%%%%%%%%%%
%%%%%%%%%%
%%%%%%%%%%
%%%%%%%%%%
if CS==1
W=input ('Enter Width of the beam in [m]: ');
Th=input ('Enter Thickness of the beam in [m]: ');
L=input ('Enter Length for the beam in [m]:
');
Ix=(1/12)*W*Th^3;
A=W*Th;
disp ('Material properties of the beam')
disp ('Do you know your beam material
properties, i.eYoung"s modulus and density ?')
YA=input('Enter 1, if you do; enter 0, if you
do not: ');
if YA==1
E=input ('Enter Young"s modulus in [Pa]:
');
Ro=input('Enter material density in
[kg/m^3]: ');
else
display ('Steel: E=2.1e+11 [Pa]; Ro=7850
[Kg/m^3] ')
display ('Copper: E=1.2e+11 [Pa]; Ro=8933
[Kg/m^3] ')
display ('Aluminum: E=0.69e+11 [Pa]; Ro=2700
[Kg/m^3] ')
display ('Zinc-oxide: E=1.11e+11 [Pa]; Ro=5680
[Kg/m^3] ')
display ('PZT: E=0.63-1.1e+11 [Pa]; Ro=7750
[Kg/m^3] ')
E=input ('Enter Young"s modulus in
[Pa]: ');
Ro=input('Enter material density in
[kg/m^3]: ');
end
elseif CS==2
display('Quick Example')
display('This is a rectangular steel beam ')
display('Length=0.45 m; Width=0.04 m;
Thickness=0.003 m;')
L=.45; W=.04; Th=.003;
A=W*Th;
Ix=(1/12)*W*Th^3;
E=2.1*1e11; Ro=7.85*1e3;
```

```

else
    F=warndlg('WARNING: INVALID ENTRY !!!','!!
Warning !!!');
waitfor(F)
display('Type in:>>Cbeam')
pause(3)
return
end
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%>>Modes<<%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
display ('How many modes and mode shapes would
you like to evaluate ?')
HMMS=input('Enter the number of modes and mode
shapes to computed: ');
if HMMS>=7
disp(' ')
warning('WARNING: Up to 6 mode shapes (plots)
are displayed via this script');
disp(' ')
end
    Nm=3*HMMS;
    jj=1;
    while jj<=Nm;
        betaNL(jj)=fzero(@(betaNL) cosh(betaNL)*cos(betaNL)+1,[jj jj+3]);
        jj=jj+3;
    end

    index=(betaNL~=0);
    betaNLall=(betaNL(index));
    fprintf('betaNL value is %2.3f\n',betaNLall);
    betaN=(betaNLall/L);
    k=1;
    wn=ones(1,length(betaN));
    fn=ones(1,length(wn));
    while k<=length(betaN);
        wn(k)=betaN(k)^2*sqrt((E*Ix)/(Ro*A));
        fn(k)=wn(k)/(2*pi);
        fprintf('Mode shape # %2f corresponds to nat.
freq (fn): %3.3f\n',k,fn(k));
        k=k+1;
    end

    x=linspace(0,L,180);
    xl=x./L;
    sigmaN=zeros(1,HMMS);
    for ii=1:HMMS;

        sigmaN(ii)=(sinh(betaN(ii)*L)-sin(betaN(ii)*L))/(cosh(betaN(ii)*L)+cos(betaN(ii)*L));
    end
    Tc='(cosh(betaN(ii).*x(jj))-cos(betaN(ii).*x(jj)))-sigmaN(ii).*(sinh(betaN(ii).*x(jj))-sin(betaN(ii).*x(jj)))';
    Xnx=zeros(length(betaN),length(x));

    for ii=1:length(betaN)
        forjj=1:length(x)
            Xnx(ii,jj)=eval(Tc);
        end
    end
    %
    % Plot mode shapes;
    %
    MMS=HMMS;
    if MMS==1
        plot(xl,Xnx(1,:), 'b-');
        title('Mode shape of the Cantilever beam')
        legend('Mode #1', 0); xlabel('x/L');
        ylabel('Mode shape X_n(x)'); grid
        holdoff
    elseif MMS==2
        plot(xl,Xnx(1,:), 'b-'); hold on
        plot(xl,Xnx(2,:), 'r-');grid
        title('Mode shapes of the Cantilever beam')
        legend('Mode #1', 'Mode #2', 0)
        xlabel('x/L'); ylabel('Mode shape X_n(x)')

```

```

holdoff;
elseif MMS==3
    plot(xl,Xnx(1,:), 'b-'); hold on
    plot(xl,Xnx(2,:), 'r-')
    plot(xl,Xnx(3,:), 'm-');grid
    title('Mode shapes of the Cantilever beam')
    legend('Mode #1', 'Mode #2', 'Mode #3', 0)
    xlabel('x/L'); ylabel('Mode shape X_n(x)')
    holdoff;
elseif MMS==4
    plot(xl,Xnx(1,:), 'b-'); hold on
    plot(xl,Xnx(2,:), 'r-')
    plot(xl,Xnx(3,:), 'm-')
    plot(xl,Xnx(4,:), 'c-'); grid
    title('Mode shapes of the Cantilever beam')
    legend('Mode #1', 'Mode #2', 'Mode #3', 'Mode #4', 0)
    xlabel('x/L'); ylabel('Mode shape X_n(x)')
    holdoff;
elseif MMS==5
    plot(xl,Xnx(1,:), 'b-'); hold on
    plot(xl,Xnx(2,:), 'r-')
    plot(xl,Xnx(3,:), 'm-')
    plot(xl,Xnx(4,:), 'g-')
    plot(xl,Xnx(5,:), 'k-')
    grid
    title('Mode shapes of the Cantilever beam')
    legend('Mode #1', 'Mode #2', 'Mode #3', 'Mode #4', 'Mode #5', 0)
    xlabel('x/L'); ylabel('Mode shape X_n(x)')
    holdoff
elseif MMS>=6
    plot(xl,Xnx(1,:), 'b-'); hold on
    plot(xl,Xnx(2,:), 'r-')
    plot(xl,Xnx(3,:), 'm-')
    plot(xl,Xnx(4,:), 'g-')
    plot(xl,Xnx(5,:), 'k-')
    plot(xl,Xnx(6,:), 'c-')
    grid
    title('Mode shapes of the Cantilever beam')
    legend('Mode #1', 'Mode #2', 'Mode #3', 'Mode #4', 'Mode #5', 'Mode #6', 0)
    xlabel('x/L'); ylabel('Mode shape X_n(x)')
    holdoff

end

end

end

```

II. Two Layer Cantilever Beam

```

function Cbeam(~)
% Cbeam.m Cantilever Beam calculations
% HELP: This script computes mode shapes and
corresponding natural for
% beam with two layers (coated beam)
%
clearall;
clc;
closeall;
display('*****')
display('*****')
display('Mohammad Abu-Shams')
display('School fo Engineering and Technology')
display('*****')
display('*****')
display('This program computes mode shapes and
crossponding natural')
display('frequencies up to 6 modes for a two
layers cantilever beam,')

```

```

display(' (the main beam (s) is coated with a thin
film layer (f)).')
display('Mechanical properties and geometry
size of the coated cantilever beam')
display('should be specified by the user')
display('Prepare to input the following
properties: Length,width,thickness(Thf),')
display('thickness(Ths),Young Modulus (Es and
Ef),density (Ros and Rof)')

display('The cross section should be
rectangular, enter 1 if this is correct, other
wise the program will be closed')
CS=input('Enter your choice: ');
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

if CS==1
    W=input('Enter Width of the beam in [m]: ');
    Ths=input('Enter Thickness of the main beam (s)
in [m]:');
    Thf=input('Enter Thickness of the thin film
layer (f) in [m]:');
    L=input('Enter Length of the whole beam in
[m]: ');
    As=W*Ths;
    Af=W*Thf;
    disp('Material properties of the beam')
    disp('Do you know your beam"s material
properties, Young"s modulus and density ?')
    YA=input('Enter 1, if you do; enter 0, if you
dont: ');
    if YA==1
        Es=input('Enter Young"s modulus in [Pa] for (s)
beam: ');
        Ros=input('Enter material density in [kg/m^3]
for (s) beam: ');
        Ef=input('Enter Young"s modulus in [Pa] for (f)
beam: ');
        Rof=input('Enter material density in [kg/m^3]
for (f) beam: ');
    else
        display('Steel: E=2.1e+11 [Pa]; Ro=7850
[Kg/m^3] ');
        display('Copper: E=1.2e+11 [Pa]; Ro=8933
[Kg/m^3] ');
        display('Aluminum: E=0.69e+11 [Pa]; Ro=2700
[Kg/m^3] ');
        display('Zinc-oxide: E=1.11e+11 [Pa]; Ro=5680
[Kg/m^3] ');
        display('PZT: E=0.63-1.1e+11 [Pa]; Ro=7750
[Kg/m^3] ');
        Es=input('Enter Young"s modulus in [Pa] for (s)
beam: ');
        Ros=input('Enter material density in [kg/m^3]
for (s) beam: ');
        Ef=input('Enter Young"s modulus in [Pa] for (f)
beam: ');
        Rof=input('Enter material density in [kg/m^3]
for (f) beam: ');
    end
else
    F=warndlg('WARNING: INVALID ENTRY !!!','!!
Warning !!!');
    waitfor(F)
    display('Type in:>>Cbeam')
    pause(3)
    return
end

y
=(Es*Ths^2+Ef*(2*Ths*Thf+Thf^2))/(2*Es*Ths)+
(2*Ef*Thf);
    Is = ((1/12)*W*(Ths^3)) +
W*Ths*(y-(Ths/2))^2;
    If= ((1/12)*W*(Thf^3)) +
W*Thf*((Thf/2)+Ths-y)^2;

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
display('How many modes and mode shapes would
you like to evaluate ?')
HMMS=input('Enter the number of modes and mode
shapes to computed: ');
if HMMS>=7
    disp(' ')
    warning('NOTE: ONLY 6 mode shapes (plots) are
displayed via the script. ');
    disp(' ')
end
    Nm=3*HMMS;
    jj=1;
    whilejj<=Nm;
        betaNL(jj)=fzero(@(betaNL) cosh(betaNL)*cos(be
taNL)+1,[jj jj+3]);
        jj=jj+3;
    end

    index=(betaNL~=0);
    betaNLall=(betaNL(index));
    fprintf('betaNL value is %2.3f\n', betaNLall);
    betaN=(betaNLall/L);
    k=1;
    wn=ones(1,length(betaN));
    fn=ones(1,length(wn));
    while k<=length(betaN);
        wn(k)=betaN(k)^2*(sqrt((Es*Is+Ef*If)/(Ros*As+
Rof*Af)));
        fn(k)=wn(k)/(2*pi);
        fprintf('Mode shape # %2f corresponds to nat.
freq (fn): %3.3f\n', k, fn(k) );
        k=k+1;
    end

    x=linspace(0, L, 180);
    xl=x./L;
    sigmaN=zeros(1, HMMS);
    for ii=1:HMMS;

        sigmaN(ii)=(sinh((betaN(ii)*L))-sin(betaN(ii)
*L))/(cosh(betaN(ii)*L)+cos(betaN(ii)*L));
    end
    Tc=' (cosh(betaN(ii).*x(jj))-cos(betaN(ii).*x(
jj)))-sigmaN(ii).*(sinh(betaN(ii).*x(jj))-sin(
(betaN(ii).*x(jj)))';
    Xnx=zeros(length(betaN),length(x));

    for ii=1:length(betaN)
        forjj=1:length(x)
            Xnx(ii,jj)=eval(Tc);
        end
    end
    %
    %Plot mode shapes;
    %
    MMS=HMMS;
    if MMS==1
        plot(xl,Xnx(1,:), 'b-')
        title('Mode shape of the coated Cantilever
beam')
        legend('Mode #1', 0); xlabel('x/L');
        ylabel('Mode shape X_n(x)'); grid
        holdoff
    elseif MMS==2
        plot(xl,Xnx(1,:), 'b-'); hold on
        plot(xl,Xnx(2,:), 'r-');grid
        title('Mode shapes of the coated Cantilever
beam')
        legend('Mode #1', 'Mode #2', 0)
        xlabel('x/L'); ylabel('Mode shape X_n(x)')
        holdoff;
    elseif MMS==3

```

```
plot(xl,Xnx(1,:), 'b-'); hold on
plot(xl,Xnx(2,:), 'r-')
plot(xl,Xnx(3,:), 'm-');grid
title('Mode shapes of the coated Cantilever
beam')
legend('Mode #1', 'Mode #2', 'Mode #3', 0)
xlabel('x/L'); ylabel('Mode shape X_n(x)')
holdoff;
elseif MMS==4
plot(xl,Xnx(1,:), 'b-'); hold on
plot(xl,Xnx(2,:), 'r-')
plot(xl,Xnx(3,:), 'm-')
plot(xl,Xnx(4,:), 'c-'); grid
title('Mode shapes of the coated Cantilever
beam')
legend('Mode #1', 'Mode #2', 'Mode #3', 'Mode
#4', 0)
xlabel('x/L'); ylabel('Mode shape X_n(x)')
holdoff;
elseif MMS==5
plot(xl,Xnx(1,:), 'b-'); hold on
plot(xl,Xnx(2,:), 'r-')
plot(xl,Xnx(3,:), 'm-')
plot(xl,Xnx(4,:), 'g-')
plot(xl,Xnx(5,:), 'k-')
grid
title('Mode shapes of the coated Cantilever
beam')
legend('Mode #1', 'Mode #2', 'Mode #3', 'Mode
#4', 'Mode #5', 0)
xlabel('x/L'); ylabel('Mode shape X_n(x)')
holdoff
elseif MMS>=6
plot(xl,Xnx(1,:), 'b-'); hold on
plot(xl,Xnx(2,:), 'r-')
plot(xl,Xnx(3,:), 'm-')
plot(xl,Xnx(4,:), 'g-')
plot(xl,Xnx(5,:), 'k-')
plot(xl,Xnx(6,:), 'c-')
grid
title('Mode shapes of the coated Cantilever
beam')
legend('Mode #1', 'Mode #2', 'Mode #3', 'Mode
#4', 'Mode #5', 'Mode #6', 0)
xlabel('x/L'); ylabel('Mode shape X_n(x)')
holdoff

end

end
```