On The Uniqueness and Solution of Certain Fractional Differential Equations

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***Abstract*— We consider the fractional differential equations with constant coefficients, using Osler definition**

$$D^{\beta} x(t) + D^{\alpha}x(t) + x(t) = 0 \qquad \text{.... (1)}$$

Where $, \beta > \alpha > 0, 0 < t < T$

In this paper we have proven that the uniqueness of solution of fractional differential equation, and solve this equation by using the Laplace transform technique.

***Index Terms*— Fractional Differential Equation, Laplace transform, Mittag-Leffler, contraction Theorem.**

I. INTRODUCTION

The field of fractional calculus (that is, calculus of integrals and derivatives of any arbitrary real or complex order) is almost as old as calculus itself.(The idea where known-Leibniz (1859) mentions in a letter to L'Hospital in (1695)) [3].

The earliest more or less systematic studies seem to have been made in the beginning and middle of the 19^{th} century by Liouville (1832a),Riemann (1847) ,and Holmgren (1864),although Euler(1730),Lagrange(1772),and others made some contributions even earlier[3] .

Through the last decades the usefulness of this mathematical theory in applications as well as its merits in pure mathematics has become more and more evident .Possibly the easiest access to the idea of the non-integer differential and integral operators studied in the field of fractional calculus is given by Cauchy's well known representation of an n-fold integral as a convolution integral [5]

$$J^n y(x) = \int_0^x \int_0^{x_{n-1}} \dots \int_0^{x_1} y(x_) \, dx_0 \dots dx_{n-2} dx_{n-1}$$

$$= \frac{1}{(n-1)!} \int_0^x \frac{1}{(x-t)^{1-n}} y(t)\, dt, \qquad n \in \mathbb{N}, x \in \mathbb{R}_{+,}$$

Where $J^n is$ the n-fold integral operator with $J^0 y(x) = y(x)$.Replacing the discrete factorial (n-1)! with Euler's continuous gamma function $\Gamma(n)$, which satisfies (n-1) $! = \Gamma(n)$ for $n \in \mathbb{N}, one\ obtains\ a\ d$ efinition of a non-integer order integral, i.e.

$$J^{\alpha} y(x) = \frac{1}{\Gamma(\alpha)} \int_0^x \frac{1}{(x-t)^{1-\alpha}} y(t)\, dt, \qquad \alpha, x \in \mathbb{R}_{+,}$$

The fact that there is obviously more than one way to define non-integer order derivatives is one of the challenging and rewarding aspects of this mathematical field.

The integral transform of a function f(x) defined in a$\leq x \leq b\ is\ denoted\ by$

$\mathscr{T}\{x(t)\} = X(t),$ and defined by

$\mathscr{T}\{x(t)\} = X(t) = \int_a^b K(t,\tau)x(t)dt,$ where,

$K(t,\tau\ ), given\ function\ of\ two\ variable\ t\ and\ \tau\ ,$

$is\ called\ the\ kernel\ of\ the\ \ transform.,$

The operator $\mathscr{T}$ is called an integral transformation.

The idea of the integral transform operator is somewhat similar to that of the well- known linear differential operator, D

$\equiv$

$\frac{d}{dt}, which\ acts\ on\ a\ function\ x(t)\ \ to\ produce\ another\ function\, x'(t), that\ is\ Dx(t) = x'(t),$

Usually,

$x'(t)\ is\ called\ the\ derivative\ or\ the\ image\ of\ x(t)\ under\ the\ linear\ transformation\ D. Evidently, there\ are\ a\ number\ of\ important\ integral\ transforms$

including Fourier, Laplace, Hankel, Mellin transforms, etc. and we chose Laplace transform.

II. PRELIMINARIES:

Many definitions of fractional derivatives and fractional integrals (fractional differintegration for short) were introduced. These definitions agree when the order is integer and some of them are different when the order is not integer. Some are equivalent such that Riemann- Liouville and Grűnwald(1867) [2]. There is an equivalent between

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Riemann-Liouvile and Osler definition when fractional order less than zero [3]. Several authors have considered different methods for calculating fractional derivatives of a given function [2 p.89- 90].
Ordinary differential equation and Integral equation is an important field of application of the contraction theorem.
Now we review some definitions and theorems.

Definition, [2]

The Laplace transform is defined as follows

If x (t) is of exponential order α and is a piece-wise continuous function Real line, then Laplace transform of x (t) for s> α

X(s) $=\mathcal{L}[x(t)] = \int_0^\infty e^{-st}\, x(t)\,dt$

And the inverse Laplace transform of X(s) is [4]

$$\mathcal{L}^{-1}\{X(s)\} = x(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st}\, x(s)\, ds$$

Note : Laplace Transform of Standard Fractional Differential Equation [2].

$$\mathcal{L}\{{}_0D_t^\alpha f(t)\} = \int_0^\infty e^{-st}\ {}_0D_t^\alpha f(t)dt = S^\alpha F(s) - \sum_{k=0}^{n} S^k D^{\alpha-k-1} f(t)\,]_{t=0}$$

$(n-1 < \alpha < n)$

Definition, [4]:

A two- parameter function of the Mittage-Leffler is defined by the series expansion

$$E_{\alpha,\beta}(z) = \sum_{K=0}^{K=\infty} \frac{Z^k}{\Gamma(\alpha k+\beta)} \qquad (\alpha > 0, \beta > 0)$$

For $\beta = 1$,we obtain

$E_{\alpha,1}(z) = \sum_{K=0}^{K=\infty} \frac{z^k}{\Gamma(\alpha k+1)} \equiv E_\alpha(z)$, is one-parameter Mittag-Leffler function

Relation between Laplace transform and Mittag-Leffler function[4 p.50].

$$[t^{\alpha n+\beta-1}\left(\frac{\partial}{\partial\lambda}\right)^n E_{\alpha,\beta}(\lambda t^\alpha)(s) = \frac{n!\, s^{\alpha-\beta}}{(s^\alpha-\lambda)^{n+1}} \quad (|\lambda s^{-\alpha}| < 1)$$

Definition: (contraction):[1]

Let X=(X, d) be a metric space .A mapping T: X$\longrightarrow X$ is called a contraction on X if there is a positive real number $0 < \alpha < 1$ such that for all x, y $\in X$

$$d(T(x), T(y)) \le \alpha\, d(x,y) \quad , \quad 0 < \alpha < 1$$

Banach Fixed point theorem(Contraction theorem [1]).

Consider a metric space X=(X, d), where X$\neq \emptyset$. *Suppose* that X is complet and let T: X$\longrightarrow X$ be a contraction on X. Then T has precisely one fixed point.

Lemma 2.1 [8].

For $\alpha \ge \beta$, $\alpha > \gamma$, a$\in \mathbb{R}$, $s^{\alpha-\beta} > |a|$ and $|s^\alpha + s^\beta| > |b|$ we have

$$L^{-1}\left[\frac{s^\gamma}{s^\alpha + as^\beta + b}\right] = x^{\alpha-\gamma-1}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-b)^n(-a)^k\binom{n+k}{k}}{\Gamma(k(\alpha-\beta)+(n+1)\alpha-\gamma)}\, x^{k(\alpha-\beta)+n\alpha}$$

3-MAIN RESULT:

In this section, we study the uniqueness of solution of our equation (1) where we proved the existence of solution of (1) by two different approaches depending on the Laplace transform technique .

Theorem 2.3.1 [Uniqueness Theorem].

Consider

$$D^\beta x(t) + D^\alpha x(t) + x(t) = 0 \qquad where,\ \beta > \alpha > 0$$

Then there exist unique solution if

$$\left|\frac{1}{2\pi}\left[\int_0^t \Gamma(\beta+1)\,(\tau-t)^{-\beta-1} + \Gamma(\alpha+1)\,(\tau-t)^{-\alpha-1}\right)d\tau\right| < 1$$

where the derivative is by means of Osler definition.

Before the proof, we need the following simplification

Assuming the conditions of lemma of (3.4) in [7] are satisfied then the solution is unique.

Suppose x(t) is a continuous function.

Since $Tx(t) = Sx(t) + x(t)$

………………………………………………………… *(2)*

And S x (t) =

$$\int_0^t [\frac{\Gamma(\beta+1)}{2\pi i}(\tau-t)^{-\beta-1} + \frac{\Gamma(\alpha+1)}{2\pi i}(\tau-t)^{-\alpha-1}]x(\tau)d\tau$$

$$Tx(t) = \int_0^t [\frac{\Gamma(\beta+1)}{2\pi i}(\tau-t)^{-\beta-1} + \frac{\Gamma(\alpha+1)}{2\pi i}(\tau-t)^{-\alpha-1}]x(\tau)d\tau + x(t) \quad \ldots\ldots\ldots.(3)$$

Let $T_* x(t) = Tx(t) - Ix(t)$

We shall show that this operator satisfies the contraction mapping theorem.

$\|T_*\, x(t) - T_* y(t)\| = \|T_*[\,x(t) - y(t)]\|$

Since $T_* x(t) = Sx(t)$ (from the relation (1),

$$= \left\|\int_0^t [\frac{\Gamma(\beta+1)}{2\pi i}(\tau-t)^{-\beta-1} + \frac{\Gamma(\alpha+1)}{2\pi i}(\tau-t)^{-\alpha-1}][x(\tau)-y(\tau)]d\tau\right\|$$

$$\leq \left|\frac{1}{2\pi}\left[\int_0^t \Gamma(\beta+1)\,(\tau-t)^{-\beta-1} + \Gamma(\alpha+1)\,(\tau-t)^{-\alpha-1}\right) d\tau\right| \,\|[x(\tau)-y(\tau)]\|$$

$$\leq \left|\frac{1}{2\pi}\left[\int_0^t \Gamma(\beta+1)\,(\tau-t)^{-\beta-1} + \Gamma(\alpha+1)\,(\tau-t)^{-\alpha-1}\right) d\tau\right| \,\|[x(t)-y(t)]\|$$

First proof :

We proved in our paper [6] that for all $t,\tau \in k(t,\tau)\,, 0 < t < \tau < T, there\ exist\ L > 0$

Such that $|k(t,\tau)| \leq L,\quad L = \frac{\Gamma(\beta+1)}{\pi}$

Therefore we can get

$\|T_* \, x(t) - T_* y(t)\| \leq L\|[x(t)-y(t)]\|$

When $L<1$, $T_* \, x(t)$ *satisfy the contraction property,*

$\|T_*[\,x(t) - y(t)]\| \leq L\|[x(t)-y(t)]\|$

$\|T_* \, z\| \leq L\|z\| \quad where\ z = [\,x(t) - y(t)]$

$T_* \, z = z \to T\ z + z = z \ \to T\ z = 2z.$

Second proof :

Also we can prove it in another approach

Let

$$\Lambda_{\alpha,\beta,t,\tau} = \frac{1}{2\pi i}\left[\int_0^t \Gamma(\beta+1)\,(\tau-t)^{-\beta-1} + \Gamma(\alpha+1)\,(\tau-t)^{-\alpha-1}\right) d\tau$$

$$= \frac{1}{2\pi i}\left[\frac{\Gamma(\beta+1)}{\beta}(-t)^{-\beta} + \frac{\Gamma(\alpha+1)}{\alpha}(-t)^{-\alpha}\right]$$

For -t complex say, $t=a+ib = re^{i\theta}$

Let r=1, then $t=e^{i\theta}$

$$\Lambda_{\alpha,\beta,t,\tau} = \frac{1}{2\pi i}\left[\frac{\Gamma(\beta+1)}{\beta r^{\beta}}(\cos\beta\theta - i\,\sin\beta\theta) + \frac{\Gamma(\alpha+1)}{\alpha r^{\alpha}}(\cos\alpha\theta - i\,\sin\alpha\theta)\right]$$

$$= \frac{-1}{2\pi}\left[\,(\Gamma(\beta)\sin\beta\theta) + (\Gamma(\alpha)\sin\alpha\,\theta)\,\right] + \frac{1}{2\pi i}\left[\,(\Gamma(\beta)\cos\beta\theta) + (\Gamma(\alpha)\ \cos\alpha\,\theta)\,\right]$$

We want $\pi < \alpha\theta\,,\ \beta\theta < 2\pi$ in order that $|A| < 1$

This is true when $\alpha=\frac{3}{2}\,,\beta=\frac{5}{2}$, as follows,

For illustration,

Suppose $\alpha=\frac{3}{2}\,,\beta=\frac{5}{2}$

$$\pi < \frac{3}{2}\,\theta \quad \to \theta > \frac{2\pi}{3} = 120$$

$$\pi < \frac{5}{2}\,\theta \quad \to \theta > \frac{2\pi}{5} = 72\,,\ we\ chose\ \theta = \frac{3\pi}{4}$$

$Let\quad -t = e^{-\frac{3\pi}{4}i} \quad (r=1,\ \theta=\frac{3\pi}{4})$

$$\Lambda_{\alpha,\beta,t,\tau} = \frac{-1}{2\pi}\left[\Gamma\left(\frac{5}{2}\right)\sin\frac{15\pi}{8} + \Gamma\left(\frac{3}{2}\right)\sin\frac{9\pi}{8}\right] + \frac{1}{2\pi i}$$

$$= \frac{-1}{2\pi}[(2.355\times -0.382) + (1.57\times -0.38)]\cdot$$

$$\left[\left(\Gamma\left(\frac{5}{2}\right)\cos\frac{15\pi}{8}\right) + \left(\Gamma\left(\frac{3}{2}\right)\cos\frac{9\pi}{8}\right)\right]$$

$$+\frac{1}{2\pi i}[(2.355\times 0.923) + (1.57\times \quad -0.92)$$

After yield simplification we get,

$|\Lambda_{\alpha,\beta,t,\tau}|<1$

It means there exist values for $\alpha\,,\beta,such\ that\ it\ is\ contraction$

$\|T_* \, x(t) - T_* y(t)\| = \|T_*[\,x(t) - y(t)]\| \leq |\Lambda_{\alpha,\beta,t,\tau}|\ \|[x(t)-y(t)]\|$

Where $|\Lambda_{\alpha,\beta,t,\tau}| < 1,$

$\|T_* \, z\| \leq |\Lambda_{\alpha,\beta,t,\tau}|\|z\| \quad where\ z = [\,x(t) - y(t)]$

It mean that there exist a unique w belongs to the domain of the operator T_* *such that*

T_* w=w → Tw – w = w → Tw=2w

III. THE SOLUTION

In this section, we obtain the solution of our equation by two methods The first one is depending on the relation between Laplace transform and Mittage-Leffler function in two parameter $E_{\alpha,\beta}(z)$ and the other by using the Laplace transform technique.

Suppose that X (t) is a sufficiently good function, i.e. Laplace transform of X(t) exists.

Our equation is

$$D^{\beta}\,x(t) + D^{\alpha}x(t) + x(t) = 0 \qquad \ldots\ldots \qquad (1)$$

Where , $\beta > \alpha > 0, 0 < t < T$

Applying the Laplace transform on equation (1), we have

$[S^{\beta} + S^{\alpha} + 1]\,L(x(t) = C \quad ,where$

$C = \sum_{k=0}^{n} D^{\beta-k-1}\,x(t)\,]_{t=0} + \sum_{r=0}^{m} D^{\alpha-r-1}\,x(t)\,]_{t=0}$

α, β *are positive number with* $\beta > \alpha$,

n is the smallest integer greater than β, $(n-1 < \beta < n)$

m is the smallest integer greater than α, $(m-1 < \alpha < m)$

$$L(x(t)) = \frac{c}{[s^{\beta}+s^{\alpha}+1]} \qquad \ldots\ldots\ldots (4)$$

Taking inverse Laplace transformation of (2), we have

$$x(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{c}{[s^{\beta}+s^{\alpha}+1]} e^{st}\, ds$$

$$= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{c}{[s^{\beta}+s^{\alpha}+1]} \sum_{\gamma=0}^{\gamma=\infty} \frac{(st)^{\gamma}}{\Gamma(\gamma+1)}\, ds$$

$$= \frac{c}{2\pi i}\sum_{\gamma=0}^{\gamma=\infty} \frac{t^{\gamma}}{\Gamma(\gamma+1)} \int_{c-i\infty}^{c+i\infty} \frac{s^{\gamma}}{[s^{\beta}+s^{\alpha}+1]}\, ds \qquad \ldots\ldots\ldots\ldots (5)$$

First method.

By using the series expansion of equation (3) to get,

$$x(t) = \frac{c}{2\pi i}\sum_{\gamma=0}^{\gamma=\infty} \frac{t^{\gamma}}{\Gamma(\gamma+1)} \int_{c-i\infty}^{c+i\infty} \frac{s^{\gamma}}{s^{\beta}+1} \frac{1}{\left[1+\frac{s^{\alpha}}{s^{\beta}+1}\right]}\, ds$$

$$= \frac{c}{2\pi i}\sum_{\gamma=0}^{\gamma=\infty} \frac{t^{\gamma}}{\Gamma(\gamma+1)} \int_{c-i\infty}^{c+i\infty} \frac{s^{\gamma}}{s^{\beta}+1} \sum_{n=0}^{n=\infty}(-1)^n \left(\frac{s^{\alpha}}{s^{\beta}+1}\right)^n ds$$

where $\left|\frac{s^{\alpha}}{s^{\beta}+1}\right| < 1$

Because of the uniformly convergence property of the infant series a few terms will attain the maximum accuracy [6], so we can write it as

$$= \frac{c}{2\pi i}\sum_{\gamma=0}^{\gamma=\infty} \frac{t^{\gamma}}{\Gamma(\gamma+1)} \int_{c-i\infty}^{c+i\infty} \frac{s^{\gamma}}{s^{\beta}+1} \sum_{n=0}^{n=k}(-1)^n \left(\frac{s^{\alpha}}{s^{\beta}+1}\right)^n ds$$

$$= \frac{c}{2\pi i}\sum_{\gamma=0}^{\gamma=\infty} \frac{t^{\gamma}}{\Gamma(\gamma+1)} \int_{c-i\infty}^{c+i\infty} \sum_{n=0}^{n=k}(-1)^n \frac{s^{n\alpha+\gamma}}{(s^{\beta}+1)^{n+1}}\, ds$$

$$= \frac{c}{2\pi i}\sum_{\gamma=0}^{\gamma=\infty} \frac{t^{\gamma}}{\Gamma(\gamma+1)} \int_{c-i\infty}^{c+i\infty} \sum_{n=0}^{n=k}(-1)^n \frac{s^{n\alpha+\gamma}}{(s^{\beta}+1)^{n+1}}\, ds$$

By the relation between Laplace and Mittag-Leffler, and

If $\beta = \delta + n\alpha + \gamma$ *then* $s^{n\alpha+\gamma} = s^{\beta-\delta}$

$$= \frac{c}{2\pi i}\sum_{\gamma=0}^{\gamma=\infty} \frac{t^{\gamma}}{\Gamma(\gamma+1)} \int_{c-i\infty}^{c+i\infty} \sum_{n=0}^{n=k}(-1)^n \frac{s^{\beta-\delta}}{(s^{\beta}+1)^{n+1}}\, ds$$

Then the inverse Laplace transform of this depending on relation [4 p.50] is

$$\frac{C}{n!} t^{\beta n+\delta-1} \left(\frac{\partial}{\partial\lambda}\right)^k E_{\beta,\delta}(-t^{\beta}). \qquad \left(\left|-s^{-\beta}\right| < 1\right)$$

Where,

$$E_{\lambda,u}^{(k)}(y) = \frac{d^k}{dy^k} E_{\lambda,u}(y) = \sum_{j=0}^{\infty} \frac{(j+k)!\, y^j}{j!\,\Gamma(\lambda j+\lambda k+u)} \quad (k = 0,1,2\ldots.)$$

It's mean that our solution is,

$$\frac{C}{n!} t^{\beta n+\delta-1} \sum_{j=0}^{\infty} \frac{(j+k)!\,(-t^{\beta})^j}{j!\,\Gamma(\beta j+\beta k+\delta)} \qquad (k = 0,1,2\ldots.)$$

$$\left(\left|-s^{-\beta}\right| < 1\right)$$

Where $\delta = \beta - n\alpha + \gamma$

Second method:

Using the Laplace transform and it's inverse depending on Lemma [2.1]

$$x(t) = \frac{c}{2\pi i}\sum_{\gamma=0}^{\gamma=\infty} \frac{t^{\gamma}}{\Gamma(\gamma+1)} \int_{c-i\infty}^{c+i\infty} \frac{s^{\gamma}}{[s^{\beta}+s^{\alpha}+1]}\, ds \qquad \ldots\ldots\ldots\ldots (5)$$

By using the series expansion

$$= \frac{c}{2\pi i}\sum_{\gamma=0}^{\gamma=\infty} \frac{t^{\gamma}}{\Gamma(\gamma+1)} \int_{c-i\infty}^{c+i\infty} \frac{s^{\gamma}}{s^{\beta}+s^{\alpha}} \frac{1}{1+\frac{1}{s^{\beta}+s^{\alpha}}}\, ds$$

$$= \frac{c}{2\pi i}\sum_{\gamma=0}^{\gamma=\infty} \frac{t^{\gamma}}{\Gamma(\gamma+1)} \int_{c-i\infty}^{c+i\infty} \frac{s^{\gamma}}{s^{\beta}+s^{\alpha}} \sum \frac{(-1)^n}{(s^{\beta}+s^{\alpha})^n}\, ds$$

$$= \frac{c}{2\pi i}\sum_{\gamma=0}^{\gamma=\infty} \frac{t^{\gamma}}{\Gamma(\gamma+1)} \int_{c-i\infty}^{c+i\infty} \sum \frac{(-1)^n s^{\gamma}}{(s^{\beta}+s^{\alpha})^{n+1}}\, ds$$

Therefore

$$= \frac{c}{2\pi i}\sum_{\gamma=0}^{\gamma=\infty} \frac{t^{\gamma}}{\Gamma(\gamma+1)} \int_{c-i\infty}^{c+i\infty} \sum \frac{(-1)^n s^{\gamma}}{\left(s^{\beta}\left(\frac{1}{s^{\beta-\alpha}}+1\right)\right)^{n+1}}\, ds$$

Using the series expansion of $\frac{1}{(1+x)^{n+1}} = \sum_{k=0}^{\infty}\binom{n+k}{k}(-x)^k$

$$= \frac{c}{2\pi i}\sum_{\gamma=0}^{\gamma=\infty} \frac{t^{\gamma}}{\Gamma(\gamma+1)} \int_{c-i\infty}^{c+i\infty} \sum \frac{(-1)^n s^{\gamma}}{s^{n\beta+\beta}} \sum_{k=0}^{\infty}\binom{n+k}{k}\left(\frac{-1}{s^{\beta-\alpha}}\right)^k ds$$

$$= \frac{c}{2\pi i}\sum_{\gamma=0}^{\gamma=\infty} \frac{t^{\gamma}}{\Gamma(\gamma+1)} \int_{c-i\infty}^{c+i\infty} \sum \frac{(-1)^n (-1)^k}{s^{n\beta}\, s^{\beta-\gamma}} \sum_{k=0}^{\infty}\binom{n+k}{k}\left(\frac{1}{s^{\beta-\alpha}}\right)^k ds$$

$$= \frac{C}{2\pi i} \sum_{\gamma=0}^{\gamma=\infty} \frac{t^{\gamma}}{\Gamma(\gamma+1)} \int_{c-i\infty}^{c+i\infty} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{\binom{n+k}{k}(-1)^{n}(-1)^{k}}{s^{\beta-\gamma}} s^{-k(\beta-\alpha)-n\beta} ds$$

and by lemma[2.1]

$$L^{-1}\left[\frac{s^{\gamma}}{s^{\beta}+s^{\alpha}+1}\right] = x^{\beta-\gamma-1} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^{n}(-1)^{k}\binom{n+k}{k}}{\Gamma(k(\beta-\alpha)+(n+1)\beta-\gamma)} x^{k(\beta-\alpha)+n\beta}$$

Therefore the solution of our equation is

$$x(t) = C\ x^{\beta-\gamma-1} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^{n}(-1)^{k}\binom{n+k}{k}}{\Gamma(k(\beta-\alpha)+(n+1)\beta-\gamma)} x^{k(\beta-\alpha)+n\beta}$$

…….. (4)

Hence (3) and (4) are two forms of the solution.
So after proving the existence of solution in previous paper using Osler definition, we show in this paper with details the uniqueness of of this solution and the form of it.

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