Estimation of Fractal Dimension of Digital Images

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Abstract— **Fractal dimension of an image can be associated with the roughness of a surface represented by a digital image. The roughness of this surface can be defined as the variation in the adjacent pixels intensities. The estimation of fractal dimension of a digital image is an important issue. A new method with higher accuracy has been sought that could be used as a fractal dimension estimator for grayscale images including digital images and remotely sensed images. Clarke's original Triangular Prism Surface Area Method has been modified considering the factors that may affect the estimated value of fractal dimension of an image to reduce the error rate in the estimation of fractal dimension.**

Index Terms— **Fractal geometry, Fractal dimension, Self-similarity, Triangular Prism Surface Area Method.**

I. INTRODUCTION

A digital image can be defined as a two dimensional grid of points called as pixels. Each pixel has its own intensity value that may differ from each other. Since each pixel may vary from its adjacent pixels, the roughness of a surface can be estimated using these pixel intensities as discussed later.

Fractals and their characteristics

The concept of fractals was first pointed out by B. Mandelbrot [1]. Before the concept of fractals, there were few questions, like, what is the shape of the natural objects such as clouds, coastlines, mountains, etc. and what is their dimension? [2]. Euclidean geometry, the only geometry known at that time, was unable to answer all these questions because natural objects cannot be categorized exactly as spheres, cones, circles etc. So, to answer all these questions, Mandelbrot suggested a new geometry called as fractal geometry which deals with the objects those are beyond the definition of traditional Euclidean geometry.

Fractals have two basic characteristics namely self-similarity or self-affinity [3]. Self-similarity is that property of a fractal object in which a small part of the object looks almost similar to the object itself. On the basis of self-similarity, a fractal object can be further classified as strictly self-similar and statistical self-similar. A fractal object is said to be strictly self-similar, if the part of that object is exactly same as the object itself; statistical self-similar, if it is similar but not exactly same. In the case of self-similarity, the scaling ratio is same in both the axes but in the case of self-affinity, the scaling ratio of the two axes could be different.

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The characteristic of strictly self-similarity can be seen in artificial objects or structures but in nature, we can analyze that nothing is strictly self-similar [4], rather, the natural objects are statistical self-similar up to a certain scale. Almost all natural objects such as mountains, clouds, coastlines, etc displays some sort of self-similarity. So, at some scale we can consider them as fractals. We can also say that a fractal structure displays a higher level of geometrical complexity [5].

If an object can be divided into 'N' equal parts, similar to the object, scaled down by a ratio 'r', then its fractal dimension can be calculated as:

$$D = \log(N) / \log(1/r) \qquad (1)$$

where D is the fractal dimension, N is the no. of parts, and r is the scaling ratio.

Fractals need to be generated by following a particular pattern retaining the self-similarity feature. In order to generate a fractal, we are given with a generator (on which we have to start) and a pattern (which we have to iterate) and what we need is to repeat the pattern over the object obtained from the previous iteration. There are a number of fractal objects generated through an iteration of given pattern, such as Koch snowflake, Sierpinski triangle, Cantor set etc [6].

One more characteristic of fractals is that the measured quantity changes with the measuring scale. This can be simply understood by the coastline paradox [1]. While measuring the length of a coastline, we will notice that as we reduce the length of the measuring scale we will get an increase in the measured length. Hence, we can say that the measured quantity is inversely proportional to the measuring scale. On the basis of these variations in the measured quantity, we can estimate something, called as fractal dimension.

Fractal Dimension

Fractal dimension is so far the only measure of a fractal object to analyze its geometrical complexity. It can be defined as a ratio of the measured quantity and measuring scale [7]. Fractal dimension, also called as Hausdorff-Besicovitch dimension, is a value of fractal objects which is fractional. The main difference between Fractal Geometry and Euclidean Geometry is that in former, the dimensions could be fractional but in later, they are strictly integers. The dimension for a line in Euclidean geometry is 1, for a plane it is 2, for a solid object it is 3. But in case of Fractal geometry, a line can acquire a dimension between 1.0 to 2.0, a plane between 2.0 to 3.0 etc. [8]. The value of fractal dimension in these objects depends upon the contortion in the objects. More the contortion of the object, higher will be its dimension.

An object is said to be fractal if and only if its Hausdorff-Besicovitch (Fractal) dimension is higher than the topological dimension of the object [9]. If the topological dimension and Hausdorff-Besicovitch dimension of an object satisfying the self-similarity feature are same then it is not considered to be a fractal object [7].
On the basis of fractal dimension, we can easily distinguish between two similar types of objects, on the basis of their geometrical complexity. Two objects with different geometrical complexity will have different fractal dimension. Higher the level of complexity of an object, greater will be its fractal dimension [10].

Applications of fractal geometry

Fractal geometry, because of its edge over the Euclidean geometry, can be used in the variety of application areas such as:

- Remote Sensing
- Medical image processing
- Telecommunication (Fractal Antenna)
- Computer graphics
- Musical Pattern generation
- Seismology
- Signal processing
- Price series analysis

The topological dimension of a digital image is believed to be 2 [7], according to the Euclidean geometry since it is a 2D surface. Fractal geometry says that there is some information which is missed by Euclidean geometry. According to fractal geometry, the dimension of a digital image will be something in between 2.0 and 3.0, depending upon the level of complexity of the image.

Fractal dimension of an image can be associated with the roughness of a surface represented by a digital image [10][11]. The roughness of a digital surface can be defined as the variation in the adjacent pixels intensities. We can say that if the variation in the adjacent pixels intensities are high then the surface will be rougher and hence its fractal dimension will also be higher than the surface with less variation in the pixel values. It is believed that the fractal dimension of a digital surface can be estimated by considering these variations into account.

II. Estimation of Fractal Dimension

The fractal dimension of a surface can be estimated by taking the pixel intensity values into account. There are a number of methods already proposed for the same such as Triangular Prism Surface Area method [12], Differential Box Counting method [13], Isarithm method [14], Variation method [14] etc.
There are some reviews on the comparison of various fractal dimension estimation methods [15][16]. Zhou and Lam [14] compared different fractal dimension estimation methods on the basis of multiple fractal surface generation algorithms. First they have generated some synthetic fractal surfaces by using the Shear displacement, Fourier filtering, and Midpoint displacement methods and then the different methods for fractal dimension estimation are applied to these surfaces. They concluded that their results clearly proved that the Triangular Prism Surface Area method and the Isarithm method are among the best estimators for fractal dimension because their RMSE was observed to be very low among all the methods.

A review article was presented by Sun et al. [17] in which the authors have reviewed most of the methods already proposed to estimate the fractal dimension of remotely sensed images. They have compared and discussed the methodologies alongwith the applications of fractal geometry in the domain of remote sensing and related areas. They have also discussed most of the issues and factors that may affect the estimation of fractal dimension.
The general procedure for most of the methods to compute the fractal dimension can be summarized as:

- Measure the quantity to be used for the estimation of fractal dimension using different measuring scales.
- A graph is to be plotted between the measured quantity and the measuring scale on log-log scale. This plot is known as Richardson plot.
- Fit a regression line over the plotted points and use its slope to estimate the fractal dimension (*D*) of the given surface.

Since present work is an improvement to Clarke's TPSAM, the same is discussed next.

Triangular Prism Surface Area Method

The Triangular Prism Surface Area Method was proposed by K.C. Clarke [12]. In his work, he proposed that the fractal dimension of a digital surface can be estimated by constructing virtual prisms over the surface using the pixel gray level values. The upper surface areas of the prisms (as shown in Figure 1) for different window sizes are thus calculated to estimate the fractal dimension of the given digital surface. On the basis of the measured quantity (i.e. the surface area) and the measuring scale (i.e. the stepsize), a log-log graph will be plotted, called as Richardson plot. A regression fit is then performed on the plotted values and the slope of that regression line will give the fractal dimension. This can be seen in Figure 2.
Thus, the fractal dimension (*D*) [12] of the surface/image can be estimated as

$$D = 2.0 - S \qquad (2)$$

where *S* represents the slope of the regression.

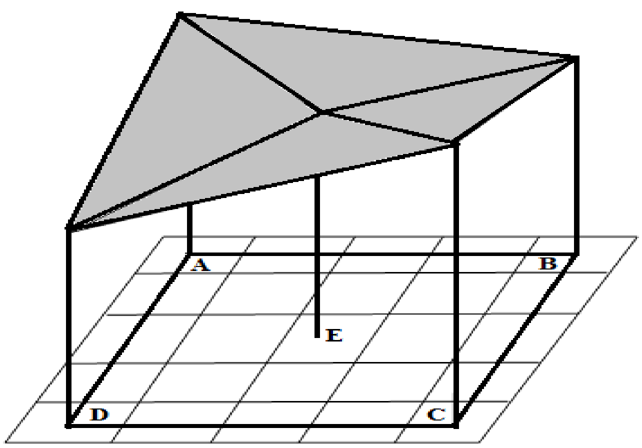


Figure 1: 3D representation of the prism constructed using the TPSAM [18].

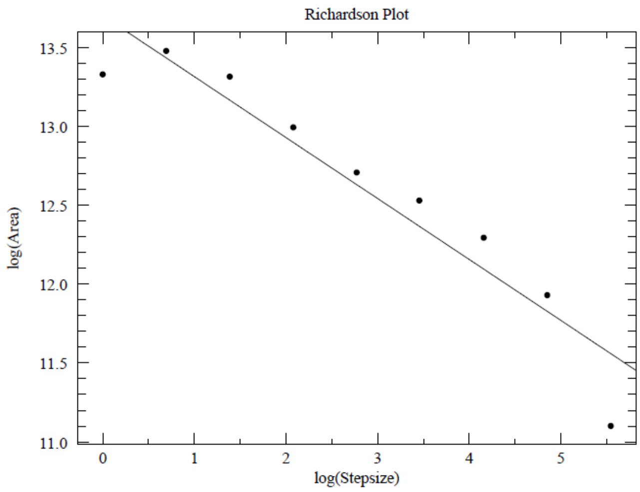


Figure 2: Richardson plot with fitted regression line.

Since the TPSAM is the most widely used method, it has been exercised in present work for finding out the robust approach to estimate the fractal dimension. A number of modifications have already been proposed for TPSAM [9][19][20].

Santis et al. [21] discussed the effect of considering prism's upper surface with (a) two triangles and (b) four triangles. The research show that although both the approaches results into similar values of areas but sometimes the prism's upper surface area calculated by considering two triangles may overestimate the values of the surface areas. They said that the values estimated by the TPSAM with four triangles estimates more accurate values of the fractal dimension.

Clarke [12] used the stepsize area i.e., suppose the stepsize is '*s'*, he used the slope of regression plotted between log (surface area) and log (s^2). Lam [22] suggested that instead of using log (s^2), we can use log (s) to obtain the slope of the regression fitting.

Clarke [12] used the geometric Stepsizes for the window because by doing this the points on the regression plot will be evenly distributed. This methodology does not guarantee that it will cover all the pixels of the given image. Ju and Lam [18] proposed an improved approach to the triangular prism surface area method to overcome this. In that work, they have proposed approaches to select the arithmetic and divisor stepsizes. They also compared their approaches with the existing geometric stepsize approach used by Clarke [12] and concluded that their approach may result into better estimation of fractal images.

Clarke [12] used only 4 corner pixels to construct the virtual prism. Sun [19] proposed that there can be more relevant pixels for construction of the 3D prism. She proposed various methods to consider the non-corner pixels while making out the prism on the basis of max-difference and mean-difference approaches. She also proposed an approach in which she considered 8 pixels to construct the prism.

Sun et al. [20] proposed one more approach to triangular prism method called as direction prism method. They said that direction dependency may significantly affect the fractal dimension estimation of a surface In direction prism method, what they consider is to diagonally calculate the surface areas under the window by rotating the window 45°. But they concluded that to validate their results they need some more research study to perform.

I. Proposed Methodology

The basic methodology to be used while implementing the fractal dimension estimator can be shown as Figure 3.

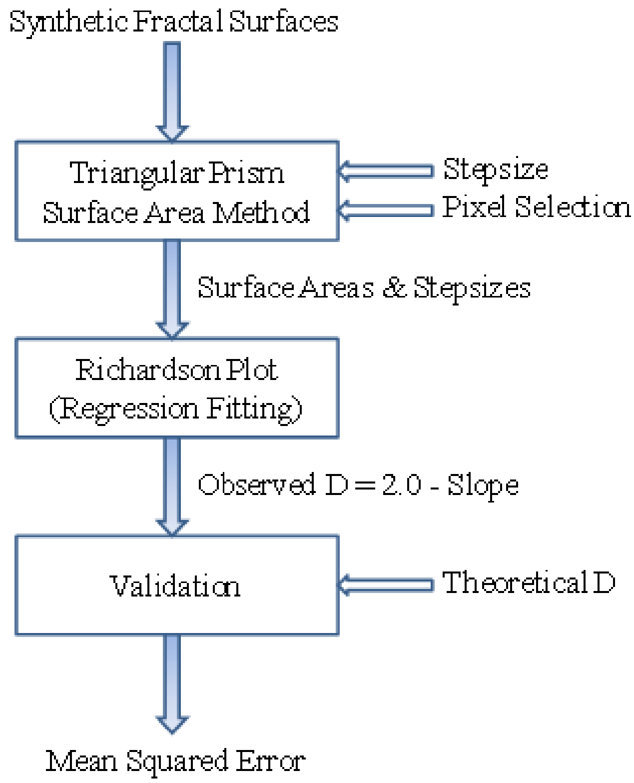


Figure 3: Methodology used during implementation.

Clarke considered the corner pixels on each edge to construct the virtual prism and on the basis of that prism he estimated the fractal dimension of digital surfaces. But apart from the corner pixels, there are a number of pixels which may affect the fractal dimension of the surface [19]. Since all those non-corner pixels may also affect the estimation of fractal dimension of a digital surface, we need to consider all of them while making out the computation. So instead of taking the corner pixels into consideration, we are proposing here to consider all corner as well as non-corner pixels of the image to estimate its fractal dimension. From each edge, choose that pixel whose deviation from the average of all pixels on the same edge is minimum because in some cases it is possible that the corner pixels have values far from the other edge pixels. In that case the effect of all non-corner pixels is nothing to do with the fractal dimension of the surface but those pixels can actually affect the fractal dimension estimation of the surface into observation.

Figure 4 and Figure 5 shows the prism and the top view of that prism respectively, if the desired pixels are not at the corner where *A*, *B*, *C*, and *D* are pixels selected on the edges and *E* is the centre pixel.

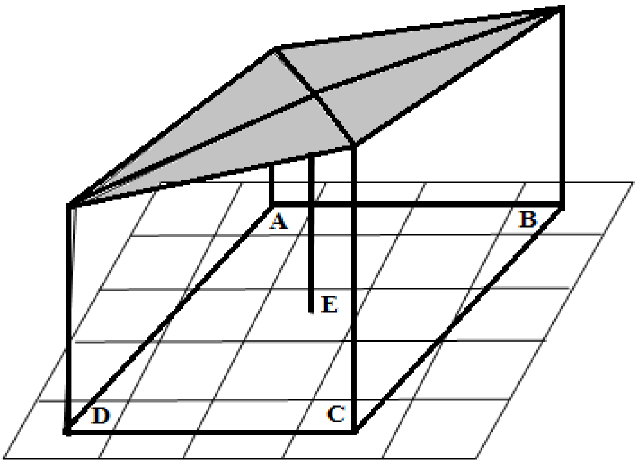


Figure 4: 3D representation of the prism constructed with non-corner pixels.

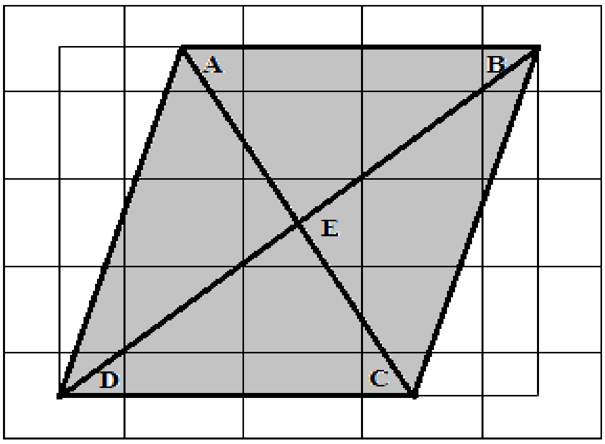


Figure 5: Top view of the constructed prism using non-corner pixels.

The value of the centre pixel '*E*' can be either calculated as an average of *A*, *B*, *C* and *D* or as actual Digital Number (*DN*) at pixel '*E*'.

On the basis of the above measured quantity, i.e. the total surface area for each stepsize and the measuring scale, i.e. the stepsize, a log-log graph will be plotted. A regression fit is then performed on the plotted values and the slope of that regression line will give the fractal dimension (*D*) as in equation (2).

The proposed methodology is implemented with few modifications on the basis of the factors that may affect the fractal dimension estimation of a digital image such as

- Overlapping and non-overlapping window sliding
- Pixel selection for constructing the prism
- Stepsizes to be used

We have implemented total 7 methods as mentioned below:

Method 1: Clarke's TPSAM.

Method 2: TPSAM with overlapping window in which the window is slided pixel by pixel.

Method 3: TPSAM with overlapping window in which the window is slided half the stepsize.

Method 4: TPSAM considering non-corner pixels for constructing the prism (as discussed in proposed methodology).

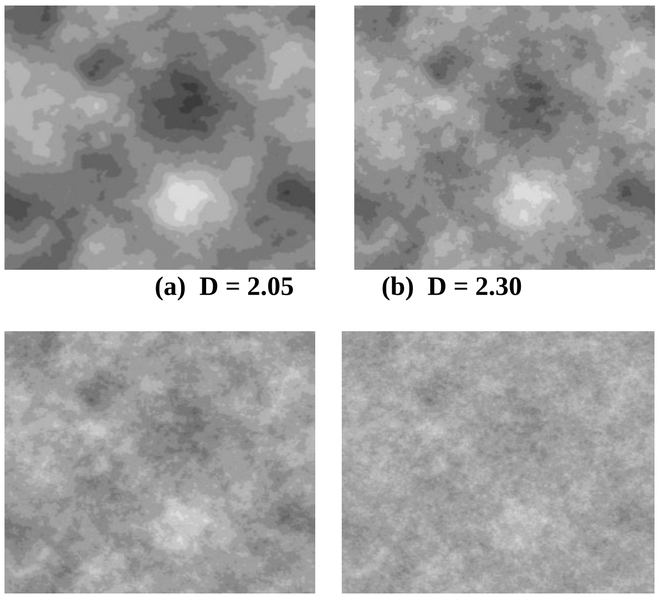
Method 5: TPSAM considering non-corner pixels for constructing the prism and using overlapping window approach.

Method 6: TPSAM considering non-corner pixels for constructing the prism and using actual value of the centre pixel.

Method 7: TPSAM considering non-corner pixels for constructing the prism and using overlapping window approach along with the centre pixel and the divisor stepsize approach

II. Results and Discussion

To verify the results of the methods mentioned in previous sections, 99 synthetic surfaces (as shown in Figure 6) generated by SynFrac [23] are used.



(a) D = 2.05 **(b) D = 2.30**

(c) D = 2.60 **(d) D = 2.90**

Figure 6: Sample synthetic fractal surfaces with known theoretical fractal dimension *D*.

The Observed value (*D'*) of fractal dimension of a particular synthetic surface is compared with its known theoretical fractal dimension (*D*). The theoretical fractal dimensions of these synthetic surfaces are already known and the measure of accuracy of the proposed methodology is the Mean Squared Error (MSE).

$$\text{MSE} = (1/n) \sum (D' - D)^2 \qquad (3)$$

The MSE of Clarke's TPSAM (Method 1) for these surfaces is 0.211523 while the MSE of the proposed methodology (Method 4) for the same set of surfaces comes out to be 0.076866 (better than Clarke's TPSAM).

The reason for doing the above mentioned experiments is to consider all the necessary parameters during the computations. The proposed methodology itself is enough to improve the results of the TPSAM but the experiments done with the proposed methodology on the basis of the factors mentioned in previous sections also comes out with positive results and the MSE for Method 7 i.e, TPSAM considering non-corner pixels for constructing the prism and using overlapping window approach along with the centre pixel and the divisor stepsize approach is lowest among all the methods mentioned in section III.

The results of Method 2 are far from accurate as shown in Figure 7. The reason behind this is the violation of the property that the measured quantity (i.e, the surface area) decreases with an increase in the measuring scale (i.e, the step size).

The comparison of all the methods is shown in Figure 7.

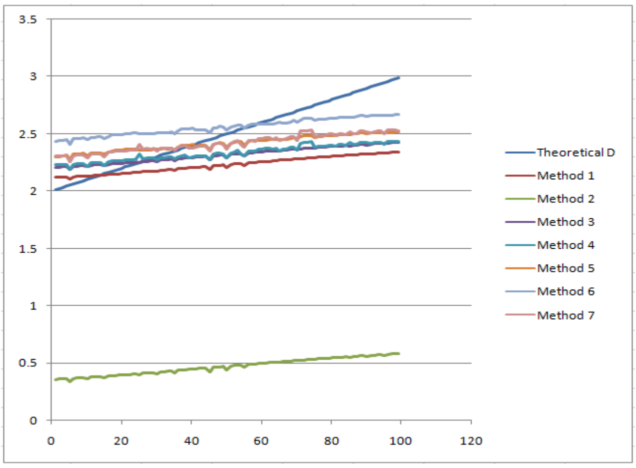


Figure 7: Comparison between all the methods mentioned in section III.

The MSE of all the methods discussed in section III is shown in Table 1.

S.No	Methods	MSE
1.	Clarke's TPSAM	0.211523
2.	TPSAM with Overlapping window (Pixel by pixel)	4.187728
3.	TPSAM with Overlapping window (Half the stepsize)	0.080556
4.	TPSAM considering non-corner pixels for constructing the prism	0.076866
5.	TPSAM considering non-corner pixels for constructing the prism and using overlapping window approach	0.056764
6.	TPSAM considering non-corner pixels for constructing the prism and using actual value of the centre pixel	0.052272
7.	TPSAM considering non-corner pixels for constructing the prism and using overlapping window approach along with the centre pixel and the divisor stepsize approach	0.051841

Table 1: Mean Squared Error of all the methods discussed in section III.

III. Conclusion and Future Work

The MSE of all the methods (as shown in Table 1) shows that for the synthetic surfaces used during the implementation, the proposed method is better than the Clarke's TPSAM. Although the estimated values are not very accurate for all the surfaces but still for most of the surfaces, the proposed method gives better results.

There are number of factors that may affect the fractal dimension estimation of a surface such as window convolution, pixel selection for prism construction, value of the centre pixel, step size etc.

Along with the above discussed factors, the accuracy of the synthetic fractal surface generator is one of the major factor that may affect the estimation of fractal dimension of a digital image. As we incorporate these factors into the computations, we will be getting better and accurate results. So, these factors need some more attention from the researchers from the related areas to figure out their proper usage while estimating the fractal dimension of an image. We need to propose a method that can result into the most accurate estimator for fractal dimension and also it must be capable enough to be applied to all the areas of digital image processing such as medical imaging, remote sensing etc.

A discussion is also required that how to estimate the fractal dimension of colored images, how the fractal dimension of separate parts of an image may affect the overall dimension of the image, i.e., the monofractal and multifractal analysis.

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