# Existence Solutions for Quasilinear Evolution Integrodifferential Equations with Infinite Delay

**Francis Paul Samuel, Tumaini RukikoLisso, Kayiita Zachary Kaunda**

**Abstract: The paper is concerned with the existence and uniqueness solution of quasi linear evolution integrodifferential equations with in finite delay in Banach spaces. The results a reobtained by $C_0$-semi group of linear bounded operator and Banach fixed point theorem.**

**Keywords: semigroup; mild andclassical solution; Banach fixed pointtheorem; Infinitedelay.**

**2000 Subject Classification:** 34G20, 47D03, 47H10,47H20,34B15.

## I. INTRODUCTION

Quasilinearevolution equations form savery important class of evolution equations as many time dependent phenomena inphysics,chemistry, biology and engineering can be represented by such evolution equations. Several authorshave studied the existence of solutions of abstract quasilinear evolution equations in Banach space [1,2,3,7,8,9,10,12,14]. Oka [10]and Okaand Tanaka[11]discussed the existence of solutions of quasilinear integrodifferential equations in Banach spaces. Kato [6] studied the non homo geneousevolution equations and Balachandran and Paul Samuel[2]proved the existence and uniqueness of mild and classical delay integro differential equation with non local condition. The problem of existence of solutions of evolution equations in Banach space has been studied by several authors [4,5,8,9].The aim of this paper is to prove the existence and uniqueness of midland classical solutions of quasilinear functional integro differential system with infinitedelay

$$u'(t) + A(t,u)u(t) = f(t,u_t) + \int_0^t k(t-s)g(s,u_s)ds \quad (1.1)$$

$$u_0 = \varphi, \text{ on } [-\mu, 0] \quad (1.2)$$

Manuscript received April 19, 2014.

**Francis Paul Samuel,** Department of Mathematics and Physics, University of Eastern Africa, Baraton,Eldoret 2500 - 30100 Kenya.
**Tumaini RukikoLisso,** ,Department of Mathematics and Physics, University of Eastern Africa, Baraton,Eldoret 2500 - 30100 Kenya.
**Kayiita Zachary Kaunda,** Department of Mathematics and Physics, University of Eastern Africa, Baraton,Eldoret 2500 - 30100 Kenya.

where$u_t(\theta) = u(t+\theta)$, $\theta \in [-?, 0]$,For $t \in [0,T]$,we denoteby$E_t$ theBanach space of all continuous functionsfrom$[-?, t]$to$X$endowed with the supremum norm

$$\|\chi\|_{E_t} = \sup_{-\tau\le\theta\le t} \|\chi(\theta)\|_X , x \in E_t.$$

Letthe functions $f: [0,T] \times E_0\ toX; g: [0,T] \times E_0$ to $X$ and $k: [0,T] \to [0,T]$ is a real value dcontinuous function. Here we see that $_t \in E_0$ ,we assume that for$u \in E_T, f(\cdot, u_{(\cdot)})$ and $g(\cdot, u_{(\cdot)}) : [0,T] \to X$are bounded $L^1$ function. Further assume that there is a subset $B$of$X$ such that for$(t,u) \in [0,T] \times E_T$with $u(t) \in B$for$t \in [0,T]$, $A(t,u)$isalinear operator in $X$.Also$\varphi \in E_0$isLipschitz continuous with constant $L_\varphi$The results obtained in this paper are generalization so the results given by Pazy [13],Kato [6,7]and Balachandranand Paul Samuel[3].

## II. PRELIMINARIES

Let $X$and $Y$ betwoBanach spaces such that Y is densely and continuous lyembedded in$X$. For any Banach spaces$Z$thenormof$Z$isdenoted by$\|\cdot\|$ or $\|\cdot\|_z$ .The space of all bounded linear operators from X to$Y$ is denoted by$B(X,Y)$and$B(X,X)$iswritten as$B(X)$.We recall some definitions and known facts from Pazy [13].

**Definition2.1.**Let$S$bealinear operatorin$X$and let$Y$ beasubspace of$X$.Theoperator$\tilde{S}$definedby $D(\tilde{S}) = \{x \in D(S) \cap Y : Sx \in Y\}$and $\tilde{S}x = Sx for\ x \in D(\tilde{S})$iscalledthepartof$S$in$Y$. $\{x + y : x \epsilon B, y \epsilon E\}$.

**Definition2.2.**Let B be a subset of $X$ and for every $0 \le t \le T$and $b \in B$ , let $A(t,b)$ be the infinitesimalgeneratorofa$C_0$ -semigroup$S_{t,b}(s), s \ge 0$, on $X$. The family of operators $\{A(t,b)\}, (t,b) \in [0,T] \times B$, isstable ifthere areconstants $M \ge 1$ and $b$ ,knownasstabilityconstants,suchthat $\rho(A(t,b)) \supset (b,\infty) f$ or $(t,b) \in [0,T] \times B$ , where $\rho(A(t,b))$ is theresolventset of $A(t,b)$ and $\left\|\prod_{j=1}^{k} R(\lambda : A(t_j, b_j))\right\| \le M(\lambda-\omega)^{-k}$for$\lambda > \omega$ every finite sequences $0 \le t_1 \le t_2 \le \ldots \le t_k \le T, b_j \in B$.

**Definition 2.3.** *Let* $S_{t,b}(s), s \ge 0$be the$C_0$- semigroup generatatedby $A(t,b), (t,b) \in I \times B$ .A subspace $Y$ of $X$ iscalled $A(t,b)$ -admissibleif $Y$ isinvariantsubspaceof$S_{t,b}(s)$andtherestrictionof$S_{t,b}(s)$to$Y$isa $C_0$-semigroup in$Y$.

Let $B \subset X$beasubset ofX such thatforevery $(t,b) \in [0,T] \times B$, $A(t,b)$ istheinfinitesimal generator ofa$C_0$-semigroup $S_{t,b}(s), s \ge 0$on$X$. Wemakethe followingassumptions:

(E1) Thefamily$\{A(t,b)\}, (t,b) \in [0,T] \times B$isstable.

(E2) $Y$ is $A(t,b)$ − admissible for $(t,b) \in [0,T] \times B$ andthe family $\{\tilde{A}(t,b)\}, (t,b) \in [0,T] \times B$ ofparts $\tilde{A}(t,b)$of$A(t,b)$in$Y$,isstable in$Y$.

(E3) For $(t,b) \in [0,T] \times B, D(A(t,b)) \supset Y, A(t,b)$ isa bounded linear operator from $Y$ to $X$and $t \to A(t,b)$iscontinuousin the$B(Y,X)$norm $\|.\|$ for every$b \in B$.

(E4) There isaconstant$L > 0$suchthat

$$\|A(t,b_1) - A(t,b_2)\|_{Y \to X} \le L\|b_1 - b_2\|_X$$

holdsforevery $b_1, b_2 \in B$ and $0 \le t \le T$. Let $B$ be a subset of $X$ and $\{A(t,b)\}, (t,b) \in [0,T] \times B$ be a family of operators satisfying the conditions (E1)-E4). If $u \in C([0,T] : X)$has value sinBthen there is a unique evolution system $U_u(t,s), 0 \le s \le t \le T$, in $X$ satisfying, (see [13,Theorem 5.3.1and Lemma 6.4.2, pp. 135,201-202]

(i) $\|U_u(t,s)\| \le Me^{\omega(t-s)}$ for $0 \le s \le t \le T$.

where M and ω are stability constants.

(ii) $\frac{\partial^+}{\partial t} U_u(t,s)w = A(s,u(s))U_u(t,s)w$ for $w \in Y$, for $0 \le s \le t \le T$.

(iii) $\frac{\partial}{\partial s} U_u(t,s)w = -U_u(t,s)A(s,u(s))w$ for $w \in Y$, for $0 \le s \le t \le T$.

(E5) For every$u \in C([0,T]: X)$satisfying $u(t) \in B$ for$0 \le t \le T$,wehave

$U_u(t,s)Y \subset Y$, $0 \le s \le t \le T$and $U_u(t,s)$isstrongly continuousin$Y$for$0 \le s \le t \le T$

(E6) Every closedconvexand bounded subsetof$Y$ is alsoclosedin$X$.

Furtherweassumethat

(E7) $f$ : $[0,T] \times E_0$to $X$ is continuousand there exist constants$F_L > 0$and $F_0 > 0$such that $\|f(t,\varphi_1) - f(t,\varphi_2)\| \le F_L(|t-s| + \|\varphi_1 - \varphi_2\|$

$$F_0 = \max \|f(t,u_o)\|.$$

(E8) $g$: $[0,T] \times E_0$to $X$ iscontinuousand thereexist constants$G_L > 0$and $G_0 > 0$such that

$$\int_0^t \|g(t,\phi_1 - g(s,\phi_2)\|_X ds \le G_L\left(|t-s| + \|\phi_1 - \phi_2\|\right)_{E_0},$$

$$G_0 = \max \int_0^t \|g(s,u_o)\| ds.$$

(E9) Thereal-valued function$k$iscontinuouson$[0,T]$ andthere existsapositiveconstant$K_T$ suchthat

$\|k(t)\| \le K_T$ for$t \in [0,T]$.Wenote thatthe condition

(E6)isalwayssatisfied if$X$and$Y$ are reflexiveBanach spaces.

Next weprovethe existence of local classical solutions of the quasilinear problem (1.1)-(1.2).

For amildsolution of(1.1)-(1.2) we mean a function $u \in E_T$with values in B satisfying the integral equation

$$u(t) = U_u(t,0)\phi(0) + \int_0^t U_u(t,s)[f(s,u_s) \quad (2.1)$$
$$+ \int_0^s k(s-\tau)g(\tau,u_\tau)]ds$$
$$u_0 = \phi \text{ on } [-\mu,0].$$

Afunction $u \in E_T$ suchthatu(t)∈Y∩Bfor t∈(0,T],u∈$C^1$((0,T]:X)and satisfies (1.1)-(1.2) in X is called[a] classical solution of(1.1)-(1.2) on[0,T], where $u \in C^1(0,T] : X)$,space of all continuously differentiable functions from $[0,T]$ to $X$ and $Y$ isa $A(t,b)$ − admissible subspace ofX.

Further there exists a constant$E_0 > 0$such that forevery $u, v \in C([0,T] : X)$with values inB and every w∈Y,wehave

$$\|U_u(t,s)w - U_v(t,s)w\| \le E_0\|w\|_Y \int_s^t \|u(\tau) - v(\tau)\| d\tau. \quad (2.2)$$

## III. Existence Results

Inthissectionwe provetheexistenceanduniqueness resultforaclassicalsolutionto(1.1)-(1.2). Let$\tilde{\varphi} \in E_T$begivenby$\tilde{\varphi}(t) = \varphi(t)$for$t \in [-\mu, 0]$and$\tilde{\varphi}(t) = \varphi(0)$for$t \in [0,T]$.Denote

$$B_r(\phi(0)) = \{x \in X : \|x - \phi(0)\|_X \le r\},$$

$$B_2r\left(\tilde{\phi}_0\right) = \{\chi \in E_0 : \left\|\chi - \tilde{\phi}_0\right\|_{E_0} \le 2r.$$

**Theorem2.1**Let $B$ and $V$ beopensubsetsof $X$ and $E_0$ respectivelyand thefamily$A(t,b)$oflinear operatorsfor$t \in [0,T], b \in B_r(\varphi(0))$ satisfyingassumptions (E1)–(E9)and$A(t,b)\varphi(0) \in Y$with

$$\|A(t,b)\phi(0)\|_Y \le C_A, \ C_A > 0$$

forall $(t,b) \in [0,T] \times B$. There existsapositiveconstant $T_0$ suchthat thequasilinearproblem(1.1)-(1.2) hasauniqueclassical solution.

$$\|U_u(t,s)\|_{B(Y)} \le K_1,$$

Take$r > 0$suchthat$B_r(\varphi(0)) \subset B$and$B_2r(\widetilde{\varphi_0}) \subset V$.

**Proof:**

Let$S$beanonemptyclosedsubset of$C([0,T]: X)$ definedby

$S = \{\psi \in C_{T_0}; \psi_0 = \varphi, for t \in [-\mu, 0], \psi(t) \in B_r(\varphi(0))\}$.

We easilydeduce that$S$is a closed, convex and bounded subsetof$C_{T_0}$. Take $\psi \in S$.Nowfor $\theta \in [-\mu, 0]$,wehavethefollowingtwocases.

Case(i): If$t + \theta \le 0$wehave

$$\left\|\psi_t(\theta)-\tilde{\phi}_0(\theta)\right\|_X=\left\|\psi(t+\theta)-\tilde{\phi}(\theta)\right\|_X$$
$$=\left\|\phi(t+\theta)-\phi(\theta)\right\|_X$$
$$=L_\phi T_0\le r.$$

Case(ii): If $t+\theta\ge 0$ we have

$$\left\|\psi_t(\theta)-\tilde{\phi}_0(\theta)\right\|_X=\left\|\psi(t+\theta)-\tilde{\phi}(\theta)\right\|_X$$
$$=\left\|\phi(t+\theta)-\phi(0)\right\|_X$$
$$=\|\phi(0)-\phi(\theta)\|_X$$
$$=r+L(-\theta)$$
$$\le r+L_\phi t$$
$$\le r+L_\phi T_0\le 2r.$$

(since $-\theta\le t\le T_0$).

Thus, for $\psi\in S$, $\psi_t\in B_{2r}(\varphi)$. Define $H:S\to S$ by

$$Hu(t)=\begin{cases}U_u(t,0)\phi(0)\\ +\int_0^t U_u(t,s)[f(s,u_s)\\ +\int_0^s k(s-\tau)g(s,u_\tau)d\tau]ds, t\in[0,T_0]\\ \phi(t), \qquad t\in[-\mu,0].\end{cases}$$

First we show that $H$ is well defined and $Hu(0)=\varphi(0)$. For $t\ge 0$, we have

$$Hu(t)-\phi(0)=U_u(t,0)\phi(0)-\phi(0)$$
$$+\int_0^t U_u(t,s)[f(s,u_s)+\int_0^s k(s-\tau)f(s,u_\tau)d\tau]ds$$

Taking the norm, we get

$$\left\|Hu(t)-\phi(0)\right\|\le\left\|U_u(t,0)\phi(0)-\phi(0)\right\|_X$$
$$+\int_0^t\left\|U_u(t,s)[f(s,u_s)+\int_0^s k(s-\tau)f(s,u_\tau)d\tau]\right\|ds$$

Integrating (ii), we obtain

$$U_u(t,0)\phi(0)-\phi(0)=\int_0^t U_u(t,s)A(s,u(s))\phi(s)ds.$$

Thus, we have

$$\left\|U_u(t,0)\phi(0)-\phi(0)\right\|_X\le\int_0^t\left\|U_u(t,s)A(s,u(s))\right\|_X\left\|\phi(s)\right\|_X ds.$$
$$\le C_A K_1 T_0$$
$$\le\frac{r}{2}.$$

Also we have

$$\int_0^t\left\|U_u(t,s)[f(s,u_s)+\int_0^s k(s-\tau)g(s,u_\tau)d\tau]\right\|ds$$
$$\le K_1\int_0^t[\left\|f(s,u_s)-f(s,u_0)+f(s,u_0)\right\|$$
$$+\left\|\int_0^s k(s-\tau)[g(\tau,u_\tau)-g(\tau,u_0)+g(\tau,u_0)]d\tau\right\|ds$$
$$\le K_1\int_0^t[\left\|f(s,u_s)-f(s,\phi)\right\|_X+\left\|f(s,u_0)\right\|$$
$$+K_T\int_0^s\left(\left\|g(\tau,u_\tau)-g(\tau,\phi)\right\|\right)d\tau]ds$$
$$\le K_1[2r(F_L+K_TG_L)+F_0+K_TG_0]T_0$$
$$\le\frac{r}{2}.$$

Using the result for $u\in S$, $u_s\in B_{2r}(\phi)$. Thus, for $u\in S$ and $t\ge 0$, we get

$$\left\|Hu(t)-\phi(0)\right\|_X\le\frac{r}{2}+\frac{r}{2}\le r.$$

So, H is well defined for $u,v\in S$, we consider

$$Hu(t)-Hv(t)=U_u(t,0)\phi(0)-U_v(t,0)\phi(0)$$
$$+\int_0^t\left\|U_u(t,s)[f(s,u_s)+\int_0^s k(s-\tau)g(s,u_\tau)d\tau]\right\|ds$$
$$-\int_0^t\left\|U_v(t,s)[f(s,v_s)+\int_0^s k(s-\tau)g(s,v_\tau)d\tau]\right\|ds$$

Let

$$I_1=\left\|U_u(t,0)\phi(0)-U_v(t,0)\phi(0)\right\|_X$$
$$\le E_0\left\|\phi(0)\right\|\int_0^t\left\|u(s)-v(s)\right\|_X ds$$
$$\le E_0\left\|\phi(0)\right\|_X\left\|u-v\right\|T_0.$$

Also let

$$I_2=\int_0^t\|U_u(t,s)[f(s,u_s)+\int_0^s k(s-\tau)[g(s,u_\tau)d\tau]$$
$$-U_u(t,s)[f(s,v_s)+\int_0^s k(s-\tau)[g(s,v_\tau)d\tau]$$
$$+U_v(t,s)[f(s,v_s)+\int_0^s k(s-\tau)[g(s,u_\tau)d\tau]ds\Big\|$$
$$\leq K_1\int_0^t\|f(s,u_s)-f(s,v_s)\|$$
$$+\int_0^s k(s-\tau)[g(s,u_\tau)-g(\tau,v_\tau)]d\tau\Big\|ds$$
$$+E_0\int_0^t[\Big\|f(s,v_s)+\int_0^s k(s-\tau)g(\tau,u_\tau)d\tau\Big\|$$
$$\times\int_s^t\|u(\tau)-v(\tau)\|_X d\tau]ds$$
$$\leq K_1[F_L\int_0^t\|u_s-v_s\|ds$$
$$+K_T\int_0^t G_L\|u_s-v_s\|ds]$$
$$+E_0[2r(F_L+K_TG_L)$$
$$+F_0K_TG_0]\|u-v\|T_0^2$$
$$\leq K_1[F_L\int_0^t\sup_\theta\|u(s+\theta-v(s+\theta)\|_X ds$$
$$+K_TG_L\int_0^t\sup_\theta\|u(s+\theta-v(s+\theta)\|_X ds$$
$$+E_0[2r(F_L+K_TG_L)+F_0+K_TG_0]\|u-v\|T_0^2$$
$$\leq K_1F_LT_0\|u-v\|+K_1K_TG_LT_0\|u-v\|$$
$$+E_0[2r(F_L+K_TG_L)+F_0$$
$$+K_TG_0]T_0^2\|u-v\|$$
$$\leq(K_1[F_L+K_TG_L]+T_0E_0[2r(F_L+K_TG_L)$$
$$+F_0+K_TG_0)T_0\|u-v\|.$$

Hence,

$$I_1+I_2=\|Hu(t)-Hv(t)\|$$
$$\leq(E_0\|\phi(0)\|_X+K_1[F_L+K_TG_T]\|$$
$$+T_0E_0[2r(F_L+K_TG_T)$$
$$+F_0+K_TG_0])T_0\|u-v\|$$
$$\leq\Omega T_0\|u-v\|_{E_{T_0}}$$
$$\leq\frac{1}{n}\|u-v\|_{E_{T_0}}.$$

Thus $H$ isacontractionfrom $S$ to $S$.So,bythe Banach contract ionmapping theorem, $H$ has aunique fixedpoint $u\in S$which satisfies the integral equation. Hence it is amild solution of(1.1)-(1.2). Now, we consider the following evolution equation

$$u'(t)+A(t,u)u(t)=f(t,u_t)+\int_0^t k(t-s)g(s,u_s)ds \quad (3.1)$$
$$u(0)=\phi(0). \quad (3.2)$$

Denote $\tilde{A}(t)=A(t,u(t))$, $\tilde{F}(t)=f(t,u_t)$ and $\tilde{G}(t)=g(t,u_t)$ds,then equation (3.1)canbewrittenas

$$u'(t)+\tilde{A}(t,u)u(t)=\tilde{F}(t)+\int_0^t k(t-s)\tilde{G}(s)ds \quad (3.3)$$
$$u(0)=\phi(0). \quad (3.4)$$

whereuistheunique solution fixedpointofH inS. Nowweassumethat(E7)-(E9)wehave

$$\|f(t,\chi)-f(s,\chi)\|_X\leq F_L|t-s|,$$
$$\int_0^t\|g(t,\chi)-g(s,\chi)\|_X ds\leq G_L|t-s|$$

and

$$\|k(t)\|\leq K_T.$$

Henceforeach∈>0there existsaδ>0suchthatif |t−s|≤δimpliesthat

$$\|f(t,\chi)-f(s,\chi)\|_X\leq\varepsilon,$$
$$\int_0^t\|g(t,\chi)-g(s,\chi)\|_X ds\leq\varepsilon.$$

Thus, $f(t,\chi)\in E_{T_0}$and $g(t,\chi)\in E_{T_0}$forfixed$\chi$.HencefromPazy[[13]Theorem 5.5.2],we getaunique function $v\in C^1((0,T_0];X)$satisfying (3.3)-(3.4) in$X$and$v$givenby

$$v(t)=U_u(t,0)\phi(0)+\int_0^t U_u(t,s)[f(s,u_s)$$
$$+\int_0^s k(s-\tau)g(s,u_\tau)d\tau]ds,\ t\in[0,T_0],$$

where$U_u(t,s),0\leq s\leq t\leq T_0$isthe evolution system generated by the family $A(t,u(t)),t\in[0,T_0]$. The uniqueness of $v$ impliesthat $v\equiv u$ on $[0,T_0]$. Thus $u$ is a unique local classicalof(1.1)-(1.2).

## References


[1] H.Amann, Quasilinearevolution equations and parabolic systems, Trans. Amer.math. Soc. 29 (1986),191-227.

[2] K.Balachandranand F.Paul Samuel, Existence ofsolutions forquasilineardelayintegrodifferentialequations withnonlocal conditions,Electron.J.Diff.Eqns.Vol.2009(2009),No.6,1-7.

[3] K.BalachandranandK.Uchiyama, Existenceof solutions of quasilinear integrodifferential equations withnonlocalcondition, Tokyo.J.Math. 23 (2000),203-210.

[4] O.Caps,Evolution equationsinscalesofBanach spaces, Verlag,TeubnerVerlag,2002.

[5] A.G.Kartsatos,The existence ofboundedsolutions onthereallineofperturbed nonlinearevolution equations ingeneral Banach spaces, Non linearAnal., 17(1991),1085-1092.

[6] S.Kato, Nonhomogeneousquasilinearevolution equationsin Banachspaces,NonlinearAnal.9 (1985),1061-1071.

[7] T. Kato, Nonlinear semigroups and evolution equations inBanach spaces,J.Math. Soc.Japan, 19(1967),508-520.

[8] T.Kato, Quasilinearequations ofevolution with applicationstopartial differential equations, Lecture NotesinMath. 448(1975),25-70.

[9] [9]T. Kato, Abstractevolution equation linear andquasilinear, revisited, Lecture Notes in Math.1540(1993),103-125.
[10] [10]H.Oka,AbstractquasilinearVolterraintegrodifferential equations, Nonlinear Anal. 28 (1997), 1019-1045.
[11] [11]H.Oka andN.Tanaka,Abstract quasilinearintegrodifferentialequations ofhyperbolictype,Non-linearAnal.29(1997),903-925.