Optimal Inventory Management for Items with Weibull Distribution Deterioration under Alternating Demand Rates

Dr. Biswaranjan Mandal

***Abstract*— The present paper deals with an inventory management for deteriorating items with instantaneous supply and alternating demand rates. A two parameters weibull distribution is taken to represent the time to deterioration and the demand is changing at a known and at a random point of time in the fixed production cycle. Finally, two numerical examples have been given for such demand fluctuations to illustrate the proposed model.**

***Index Terms*— Inventory, deterioration, time epoch, random oint of time,**

AMS Subject Classification: 90B05

I. Introduction

The economic order quantity model is the oldest inventory management model. In 1915, Harris[1] has discussed this model. Ostenyoung et al[2], in one of their research publications have reported that this model widely used in industry, although the assumptions necessary to justify its use such as constant demand rate, known inventory holding and set up costs, no shortages allowed etc. are rarely met.

In many inventory models, the effect of deterioration is very important. Ghare and Schrader[3] developed a model assuming a constant rate of deterioration. The assumption of constant deterioration rate was relaxed by Covert and Philip[4] who used a two parameter weibull distribution to represent the distribution of the time to deterioration. The model was further generalized by Philip[5] by taking a three parameter weibull distribution. Misra[6] also adopted a two parameter weibull distribution deterioration to develop an inventory model eith finite rate of replenishment. In this connection, the works done by Jalan et al[7], Dixit and Shah[8], Gupta and Agarwal[9], Cohen[10], Hwang[11] are note worthy.

An extensive research work has already been done by many researchers on inventory model by assuming a constant demand rate. Moreover the time dependent demand rates have been investigated by various researchers like Donaldson[12], Silver[13], Ritchie[14], Datta and Pal[15], Mandal and Pal[16] etc. But Dhandra and Prasad[17] have discussed an inventory model assuming demand rates for the inventory changing at a known and at a random point of time in the fixed production cycle. Pal and Mandal[18] have extended this work to a situation where the inventory is deteriorating at a constant rate. Such fluctuation in the demand rate is quite realistic and it may occur due to any of the following reasons

(i). a particular item becoming obsolete day by day and non-
availability of its spare parts

(ii). Due to higher or lower purchasing cost of items.

In the present paper, we derive an inventory model of items that deteriorate at a weibull distributed rate, assuming the alternating demand rates. Moreover the model has been developed for both the deterministic and stochastic situations. Some particular cases have been discussed and some numerical examples have been provided for such both demand pattern to illustrate the present model.

II. Preliminaries

Assumptions:

(i) Lead time is negligible.
(ii) Replenishment size is constant and its rate is infinite.
(iii) Shortages are not allowed.
(iv) A deteriorating item is neither repaired nor replaced during a given cycle.
(v) Length of each production cycle is constant.

Notations:

T : Length of one cycle.
I(t) : Inventory level at time $t(\geq 0)$.
R(t) : Demand rate at time $t(\geq 0)$.
Y : Time epoch at which the demand rate changes.
K : Fixed ordering cost of inventory.
c : Cost of unit item.
h : Holding cost per unit per unit time.
$\theta(t)$: The deterioration rate function for a two parameter weibull distribution.

III. Mathematical Formulation and Solutions

Let Q be the total amount of inventory produced at the beginning of each period. Since we have assumed an alternating demand rate for the cycle time T, we can consider the demand rate R(t) in the following form:

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Dr. Biswaranjan Mandal, Department of Mathematics, A.J.C. Bose College, Kolkata

$$R(t) = \begin{cases} \lambda_1, & 0 \le t \le z \\ \lambda_2, & z \le t \le T \end{cases}$$

where z=min(Y,T) is the epoch at which the demand rate changes.

The deterioration rate θ(t) is taken as

$$\theta(t) = \alpha\beta\, t^{\beta-1},\ 0<\alpha<<1,\ \beta \ge 1, t \ge 0.$$

Hence the differential equations governing the system in the interval (0,T) are the following

$$\frac{dI(t)}{dt} + \alpha\beta\, t^{\beta-1} I(t) = -\lambda_1,\ 0 \le t \le z \qquad (1)$$

and $$\frac{dI(t)}{dt} + \alpha\beta\, t^{\beta-1} I(t) = -\lambda_2,\ z \le t \le T \qquad (2)$$

The boundary conditions are I(0) = Q and I(T) = 0 (3)

When 0<α<<1, we ignore the terms of O(α^2), and use the condition I(0) = Q , then the solutions of the equations (1) and (2) are respectively the following

$$I(t) = Q(1 - \alpha t^{\beta}) - \lambda_1 [\, t - \frac{\alpha\beta}{\beta+1} t^{\beta+1}],\ 0 \le t \le z \qquad (4)$$

and $$I(t) = Q(1 - \alpha t^{\beta}) - (\lambda_1 - \lambda_2)[\, z(1 - \alpha t^{\beta}) + \frac{\alpha}{\beta+1} z^{\beta+1}]$$

$$- \lambda_2 [\, t - \frac{\alpha\beta}{\beta+1} t^{\beta+1}],\ z \le t \le T \qquad (5)$$

Using the condition I(T) = 0 and neglecting the terms of O(α^2), we have

$$Q = (\lambda_1 - \lambda_2)[\, z + \frac{\alpha}{\beta+1} z^{\beta+1}] + \lambda_2 [\, T + \frac{\alpha}{\beta+1} T^{\beta+1}] \quad (6)$$

The average number of units AI(T) in the inventory during the period (0,T) is

$$AI(T) = \frac{1}{T}\int_0^T I(t)dt = \frac{1}{T}[\int_0^z I(t)dt + \int_z^T I(t)dt]$$

Substituting the values of I(t) in the above integrals and then eliminating Q using equation(6) and neglecting higher order terms of α we find the following

$$AI(T) = \frac{1}{T}[(\lambda_1 - \lambda_2)\{\frac{z^2}{2} + \frac{\alpha\beta}{(\beta+1)(\beta+2)} z^{\beta+2}\} + \lambda_2 \{\frac{T^2}{2} + \frac{\alpha\beta}{(\beta+1)(\beta+2)} T^{\beta+2}\}] \quad (7)$$

Therefore, the average total cost per unit time in (0,T) is

$$C(T, z) = \frac{K + cQ}{T} + \frac{h}{2T}[(\lambda_1 - \lambda_2) z^2 \{1 + \frac{2\alpha\beta}{(\beta+1)(\beta+2)} z^{\beta}\}$$

$$+ \lambda_2 T^2 \{1 + \frac{2\alpha\beta}{(\beta+1)(\beta+2)} T^{\beta}\}]$$

$$= \frac{K}{T} + \frac{c}{T}[(\lambda_1 - \lambda_2) z\{1 + \frac{\alpha}{\beta+1} z^{\beta}\} + \lambda_2 T\{1 + \frac{\alpha}{\beta+1} T^{\beta}\}]$$

$$+ \frac{h}{2T}[(\lambda_1 - \lambda_2) z^2 \{1 + \frac{2\alpha\beta}{(\beta+1)(\beta+2)} z^{\beta}\} + \lambda_2 T^2 \{1 + \frac{2\alpha\beta}{(\beta+1)(\beta+2)} T^{\beta}\}] \quad (8)$$

We now minimize the above total average cost per unit time under the following two cases
(i). z is a known point of time
and (ii). z is a random point of time.

***Case-1**: z is a known point of time.*

(a) When z ≥ T.

In this case, the expression for C(T, z) reduces to the following

$$C(T) = \frac{K}{T} + c\lambda_1 [\, 1 + \frac{\alpha}{\beta+1} T^{\beta}] + \frac{h\lambda_1}{2} T[\, 1 + \frac{2\alpha\beta}{(\beta+1)(\beta+2)} T^{\beta}]$$

which is the cost function of the economic order quantity inventory model for deteriorating items.

If there be no deterioration i.e. α = 0, the above expression becomes

$$C(T) = \frac{K}{T} + c\lambda_1 + \frac{h\lambda_1}{2}T$$

which is similar to the equation obtained by Harris and Wilson.

(b) When z<T.

We consider z = μT, 0< μ < 1.
Thus from expression(7), we find

$$AI(T) = \frac{1}{T}[(\lambda_1 - \lambda_2)\mu^2\{\frac{T^2}{2} + \frac{\alpha\beta\mu^\beta}{(\beta+1)(\beta+2)}T^{\beta+2}\} + \lambda_2\{\frac{T^2}{2} + \frac{\alpha\beta}{(\beta+1)(\beta+2)}T^{\beta+2}\}]$$

Therefore the equation (8) reduces to the following

$$C(T) = \frac{K}{T} + \frac{c}{T}[(\lambda_1 - \lambda_2)\mu T\{1 + \frac{\alpha\mu^\beta}{\beta+1}T^\beta\} + \lambda_2 T\{1 + \frac{\alpha}{\beta+1}T^\beta\}]$$

$$+ \frac{h}{2T}[(\lambda_1 - \lambda_2)\mu^2 T^2\{1 + \frac{2\alpha\beta\mu^\beta}{(\beta+1)(\beta+2)}T^\beta\} + \lambda_2 T^2\{1 + \frac{2\alpha\beta}{(\beta+1)(\beta+2)}T^\beta\}] \quad (9)$$

For minimum average cost, the necessary condition is $\frac{\partial C(T)}{dT} = 0$

This gives $\frac{h\alpha\beta}{\beta+2}T^{\beta+2}[\lambda_1\mu^{\beta+2} + \lambda_2(1-\mu^{\beta+2})] + \frac{c\alpha\beta}{\beta+1}T^{\beta+1}[\lambda_1\mu^{\beta+1} + \lambda_2(1-\mu^{\beta+1})]$

$$+ \frac{h}{2}T^2[\lambda_1\mu^2 + \lambda_2(1-\mu^2)] - K = 0 \quad (10)$$

Let T^* be the positive real root of the above equation (10), then T^* is the optimum cycle time.

It can also be seen that the sufficient condition for the minimum cost $\frac{\partial^2 C(T)}{\partial T^2}\Big|_{T=T^*} > 0$ would be satisfied.

From expression (6), we have the optimum value of Q as follows

$$Q^* = (\lambda_1 - \lambda_2)\mu T^*[1 + \frac{\alpha\mu^\beta}{\beta+1}T^{*\beta}] + \lambda_2 T^*[1 + \frac{\alpha}{\beta+1}T^{*\beta}] \quad (11)$$

The optimum average total cost of C(T) is C(T^*).

Some particular cases

(i). ***Absence of deterioration***:

If the deterioration of items is switched off i.e. α = 0, then the expressions of C(T^*) and the optimum ordering quantity Q^* become

$$C(T^*) = \frac{K}{T^*} + c[\lambda_1\mu + \lambda_2(1-\mu)] + \frac{h}{2}T^*[\lambda_1\mu^2 + \lambda_2(1-\mu^2)]$$

and $Q^* = T^*[\lambda_1\mu + \lambda_2(1-\mu)]$

where $T^* = \sqrt{2K/h[\lambda_1\mu^2 + \lambda_2(1-\mu^2)]}$

(ii). ***The deterioration rate is constant i.e. β = 1*.**

In this case, C(T^*) and Q^* are given by the following

$$C(T^*) = \frac{K}{T^*} + \frac{c}{T^*}\,[(\lambda_1 - \lambda_2)\mu T^* \{1 + \frac{\alpha\mu}{2}T^*\} + \lambda_2 T^* \{1 + \frac{\alpha}{2}T^*\}]$$

$$+ \frac{h}{2T^*}[(\lambda_1 - \lambda_2)\mu^2 T^{*2}\{1 + \frac{\alpha\mu}{3}T^*\} + \lambda_2 T^{*2}\{1 + \frac{\alpha}{3}T^*\}]$$

which is the cost function of the inventory model obtained by Pal and Mandal[18].

$$\text{and } Q^* = (\lambda_1 - \lambda_2)\,\mu T^* [1 + \frac{\alpha\mu}{2}T^*] + \lambda_2 T^* [1 + \frac{\alpha}{2}T^*]$$

where the real positive value of T^* is obtained by the following cubic equation in T

$$\frac{h\alpha}{3}T^3[\lambda_1\mu^3 + \lambda_2(1-\mu^3)] + \frac{(c\alpha + h)}{2}T^2[\lambda_1\mu^2 + \lambda_2(1-\mu^2)] - K = 0$$

(iii). ***The demand rate is uniform(*** $\lambda_1 = \lambda_2$ ***):***

Here the expression for minimum average cost C(T^*) and the optimum order quantity Q^* are found to be

$$C(T^*) = \frac{K}{T^*} + c\lambda_1[1 + \frac{\alpha}{\beta+1}T^{*\beta}] + \frac{h\lambda_1}{2}T^*[1 + \frac{2\alpha\beta}{(\beta+1)(\beta+2)}T^{*\beta}]$$

$$\text{and } \quad Q^* = \lambda_1 T^*[1 + \frac{\alpha}{\beta+1}T^{*\beta}]$$

where the optimum cycle time T^* is the positive real root of the equation

$$\frac{h\alpha\beta\lambda_1}{\beta+2}T^{\beta+2} + \frac{c\alpha\beta\lambda_1}{\beta+1}T^{\beta+1} + \frac{h\lambda_1}{2}T^2 - K = 0$$

(iv). ***Uniform demand rate and absence of deterioration(*** $\lambda_1 = \lambda_2$; $\alpha = 0$ ***):***

In this case, the expression for the minimum cost and ordering quantity are the following

$$C(T^*) = c\lambda_1 + \sqrt{2hK\lambda_1} \quad \text{and} \quad Q^* = \sqrt{2K\lambda_1 / h}$$

where the optimum cycle time T^* is given by

$$T^* = \sqrt{2K / h\lambda_1}$$

Case-2: z is a random point of time.

From the expression (8), we have

$$C(T, z) = \frac{K}{T} + \frac{c}{T}[(\lambda_1 - \lambda_2)z\{1 + \frac{\alpha}{\beta+1}z^\beta\} + \lambda_2 T\{1 + \frac{\alpha}{\beta+1}T^\beta\}]$$

$$+ \frac{h}{2T}[(\lambda_1 - \lambda_2)z^2\{1 + \frac{2\alpha\beta}{(\beta+1)(\beta+2)}z^\beta\} + \lambda_2 T^2\{1 + \frac{2\alpha\beta}{(\beta+1)(\beta+2)}T^\beta\}] \quad (12)$$

Here the above cost function is a random variable with respect to z.
Therefore the expected total cost per unit time becomes

$$\xi(T) = \frac{K}{T} + \frac{c}{T}[(\lambda_1 - \lambda_2)\{E(z) + \frac{\alpha}{\beta+1}E(z^{\beta+1})\} + \lambda_2 T\{1 + \frac{\alpha}{\beta+1}T^{\beta}\}] + \frac{h}{2T}[(\lambda_1 - \lambda_2)\{E(z^2) + \frac{2\alpha\beta}{(\beta+1)(\beta+2)}E(z^{\beta+2})\} + \lambda_2 T^2\{1 + \frac{2\alpha\beta}{(\beta+1)(\beta+2)}T^{\beta}\}] \quad (13)$$

Let us assume the distribution function of z to be $F_z(\cdot)$.

Therefore we can take

$\overline{F_z}(u) = \begin{cases} F_Y(u), & u<T \\ 0, & u\geq T \end{cases}$ as in Dhandra and Prasad [17].

Hence the expected values of z, $z^2, z^{\beta+1}$ and $z^{\beta+2}$ are the following

$$E(z) = -\int_0^T \overline{F_Y}(u)du, \quad E(z^2) = -\int_0^T 2u\overline{F_Y}(u)du$$

$$E(z^{\beta+1}) = -\int_0^T (\beta+1)u^{\beta}\overline{F_Y}(u)du \text{ and } E(z^{\beta+2}) = -\int_0^T (\beta+2)u^{\beta+1}\overline{F_Y}(u)du.$$

Substituting the above integrals in the equation (13), the expected total cost reduces to the following

$$\xi(T) = \frac{1}{T}[K - (\lambda_1 - \lambda_2)\{c\int_0^T \overline{F_Y}(u)du + h\int_0^T u\overline{F_Y}(u)du + c\alpha\int_0^T u^{\beta}\overline{F_Y}(u)du + \frac{h\alpha\beta}{\beta+1}\int_0^T u^{\beta+1}\overline{F_Y}(u)du\}] + \frac{\lambda_2}{T}[cT + \frac{h}{2}T^2 + \frac{c\alpha}{\beta+1}T^{\beta+1} + \frac{h\alpha\beta}{(\beta+1)(\beta+2)}T^{\beta+2}] \quad (14)$$

The equation (14) is of the form

$$\xi(T) = \frac{R+\psi(T)}{\phi(T)}, \ T \geq 0 \text{ where } R=K.$$

Now $\phi(0) = \psi(0)$, and $\phi(T)$, $\psi(T)$ are continuously differential functions in [0, ∞) with strictly positive derivatives $\phi'(T)$ and $\psi'(T)$ for all $T \geq 0$.

Moreover $D(T) = \frac{\psi'(T)}{\phi'(T)}$

$$= [\lambda_2 + (\lambda_2 - \lambda_1)\overline{F_Y}(T)][c + hT + c\alpha T^{\beta} + \frac{h\alpha\beta}{\beta+1}T^{\beta+1}]$$

is a monotonic increasing function of T for $\lambda_2 > \lambda_1$.

Hence by using Puri and Singh's[19] result, we get that there exists unique global minimum for a value of T which is the optimum cycle time T^*.

Putting T=T^* in the expression ((14), the minimum expected total cost can be evaluated.

If $\lambda_2 < \lambda_1$, we assume the time instant z to be a function of some parameter 'γ' which ranges over some space 'Γ' in which a probability density function 'p(γ)' is defined such that $\int_{\Gamma} p(\gamma)d\gamma = 1$.

Therefore the expected total cost $\xi(T)$ per unit time can be written as follows

$$\xi(T) = \frac{1}{T}[K + c(\lambda_1 - \lambda_2)\{\int_{\Gamma} z(\gamma)p(\gamma)d\gamma + \frac{\alpha}{\beta+1}\int_{\Gamma} z^{\beta+1}(\gamma)p(\gamma)d\gamma\} + c\lambda_2 T(1 + \frac{\alpha}{\beta+1}T^{\beta})]$$

$$+ \frac{h}{2T}[(\lambda_1 - \lambda_2)\{\int_{\Gamma} z^2(\gamma)p(\gamma)d\gamma + \frac{2\alpha\beta}{(\beta+1)(\beta+2)}\int_{\Gamma} z^{\beta+2}(\gamma)p(\gamma)d\gamma\}$$

$$+ \lambda_2 T^2 \{ 1 + \frac{2\alpha\beta}{(\beta+1)(\beta+2)} T^{\beta} \}] \quad (15)$$

From (6), the expected ordering quantity becomes

$$<Q> = (\lambda_1 - \lambda_2)[\int_{\Gamma} z(\gamma)p(\gamma)d\gamma + \frac{\alpha}{\beta+1}\int_{\Gamma} z^{\beta+1}(\gamma)p(\gamma)d\gamma] + \lambda_2 T(1+\frac{\alpha}{\beta+1}T^{\beta})] \quad (16)$$

If the random variable z(γ) and probability density function p(γ) are prescribed, then the right hand side of the equation (15) can be evaluated.

The necessary condition for the minimum expected total cost $\xi(T)$ per unit time is $\frac{\partial \xi(T)}{\partial T} = 0$.

This gives the optimum value of T= T^* (>0).

For T= T^* , the sufficient condition for minimum $\frac{\partial^2 \xi(T)}{\partial T^2} > 0$ would be satisfied.

Then the optimum expected ordering quantity of <Q> i.e. $< Q^* >$ can be obtained from the expression (16).

IV. Numerical Examples

Example-1[Deterministic demand pattern i.e. z=known point of time]

Let the parameters of the inventory model be as follows:

h = \$ 3 per unit per year, c = \$ 5 per unit , K=\$ 20 per order, α = 0.002,
β = 2, λ_1 = 10 units, λ_2 = 5 units, μ = 0.2.

Based on these input data, the optimal values of cycle time, ordering quantity and minimum total average cost are found to be

T^* = 1.592 years, Q^* = 9.57 units and C(T^*) = \$ 55.04 per year.

Example-2[Stochastic demand pattern i.e. z=random point of time]

Let z(γ) = a + b γ , a, b>0 and the probability density function p(γ) be defined as follows

$$p(\gamma) = \begin{cases} \gamma \exp(-\gamma), & \gamma \geq 0 \\ 0, & elsewhere \end{cases}$$

Assuming a = 0.2 and b = 0.1 and retaining the values of the other parameters unchanged(as in example-1), on solving the equation $\frac{\partial \xi(T)}{dT} = 0$ where $\xi(T)$ is given by (15), we find the optimum mean value of T is 2.035 years.

From (16) and (15), and using the above mean value of T, the optimum expected ordering quantity $< Q^* >$ and minimum expected total cost $\xi(T^*)$ are 12.20 units and \$ 55.74 per year.

V. References

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