Ten Problems of Number Theory

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Abstract—In this paper we will show solution of ten problems in number theory.

Problem 1: Balanced Primes are infinite.

Problem 2: Euclid primes are infinite.

Problem 3: Quasiperfect number doesn’t exist.

Problem 4: Prime triplets are infinite.

Problem 5: Odd Superperfect number doesn’t exist.

Problem 6: Proof of Polignac’s conjecture.

Problem 7: There are infinitely many prime of the form $n^2+1$.

Problem 8: Lonely runner conjecture proof when velocities are in arithmetic progression.

Problem 9: We will find the value of imaginary part of non-trivial zeros of Riemann zeta function.

Problem 10: We will prove Landau, Parkin and Selfridge conjecture.

Index Terms—Balanced Primes, Euclid primes, Prime triplets, Polignac’s conjecture.

I. Problem 1:

A balanced prime is a prime number that is equal to the arithmetic mean of the nearest primes above and below. Or to put it algebraically, given a prime number $p_n$, where $n$ is its index in the ordered set of prime numbers,

$$p_n = \frac{p_{n-1} + p_{n+1}}{2}.$$

The first few balanced primes are


For example, 53 is the sixteenth prime. The fifteenth and seventeenth primes, 47 and 59, add up to 106, half of which is 53, thus 53 is a balanced prime.

When 1 was considered a prime number, 2 would have correspondingly been considered the first balanced prime since

$$2 = \frac{1 + 3}{2}.$$

It is conjectured that there are infinitely many balanced primes.


Solution

Let there are finite number of balanced primes.

We are considering primes of the form $p-12$, $p-6$, $p$.

Now let’s $p_n - 6$ be the last balanced prime.

Now we form odd numbers as below:

- Red coloured numbers are divisible by 3.
- Blue coloured numbers are divisible by 5.
- Green coloured numbers are divisible by 7.

$p_n - 10$, $p_n - 8$, $p_n - 6$, $p_n - 4$, $p_n - 2$, $p_n$,

$p_n + 2$, $p_n + 4$, $p_n + 6$,

$p_n + 8$, $p_n + 10$, $p_n + 12$,

$p_n + 14$, $p_n + 16$, $p_n + 18$,

$p_n + 20$, $p_n + 22$, $p_n + 24$,

$p_n + 26$, $p_n + 28$, $p_n + 30$,

$p_n + 32$, $p_n + 34$, $p_n + 36$,

$p_n + 38$, $p_n + 40$, $p_n + 42$,

$p_n + 44$, $p_n + 46$, $p_n + 48$,

$p_n + 50$, $p_n + 52$, $p_n + 54$,

$p_n + 56$, $p_n + 58$, $p_n + 60$,

$p_n + 62$, $p_n + 64$, $p_n + 66$,

$p_n + 68$, $p_n + 70$, $p_n + 72$,

$p_n + 74$, $p_n + 76$, $p_n + 78$,

$p_n + 80$, $p_n + 82$, $p_n + 84$,

$p_n + 86$, $p_n + 88$, $p_n + 90$,

$p_n + 92$, $p_n + 94$, $p_n + 96$,

$p_n + 98$, $p_n + 100$, $p_n + 102$,

$p_n + 104$, $p_n + 106$, $p_n + 108$.

We see that $p_n + 96$ and $p_n + 102$ are composite and forming a balanced prime sets viz. $p_n + 94$, $p_n + 100$ and $p_n + 106$.

Now according to our assumption one of this must be divisible by any prime before it i.e. one of them have to be composite.

Now, we will find infinite number of balanced prime sets only by 5 and 7 and it will occur in a regular frequency i.e. after every 70 numbers and in a particular column it will occur after every 210 numbers.

There are only finite number of primes before $p_n$.

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They cannot make every balanced prime set unbalanced by dividing any of three primes forming the balanced prime set.

So, there needs to be born primes after $p_k$, so that every regular occurrence of balanced prime set by 5,7 can be made unbalanced by dividing with those primes at least one of the prime forming balanced set. But there is no regular occurrence of prime whereas 5,7 will be forming balanced set of primes in a regular manner.

There will be a shortcoming somewhere of primes to make each and every balanced pair formed by 5,7 in a regular manner.

There will be balanced prime after $p_n$.

Here is the contradiction.

Balanced primes are infinite.

II. Problem 2:

In mathematics, Euclid numbers are integers of the form $E_n = p_n# + 1$, where $p_n#$ is the nth primorial, i.e. the product of the first $n$ primes. They are named after the ancient Greek mathematician Euclid.

The first few Euclid numbers are 3, 7, 31, 211, 2311, 30031, 510511 (sequence A006862 in OEIS).

It is not known whether or not there are an infinite number of prime Euclid numbers.

$E_6 = 13# + 1 = 30031 = 59 \times 509$ is the first composite Euclid number, demonstrating that not all Euclid numbers are prime. A Euclid number cannot be a square. This is because Euclid numbers are always congruent to 3 mod 4.

For all $n \geq 3$ the last digit of $E_n$ is 1, since $E_n - 1$ is divisible by 2 and 5.

Problem source: http://en.wikipedia.org/wiki/Euclid_number

Solution

Let Euclid prime is finite.

Let the last Euclid prime is $E_n = p_n# + 1 = p_1 p_2 \ldots p_n + 1$.

Now, $E_n = p_n# + 1 = p_1 p_2 \ldots p_n + 1 \equiv 1$ (mod 3) and $-1$ (mod 4)

$E_n \equiv 12p + 7$ form.

$E_n \equiv 3$ (mod 8) (considering p as odd)

Now there will be cases when $p_1 p_2 \ldots p_n \equiv \pm 2$ (mod 8).

There will be cases when $p_1 p_2 \ldots p_n \equiv -2$ (mod 8).

$p_1 p_2 \ldots p_n + 1 \equiv 7$ (mod 8) which is contradiction.

$p$ can be even.

Putting 2p in place of p we get, $E_k = 24p + 7$ (p is odd)

Now, $24p + 7 \equiv -1$ (mod 16)

Now, $p_1 p_2 \ldots p_k \equiv \pm 2, \pm 6$ (mod 16)

$\Rightarrow p_1 p_2 \ldots p_k + 1 \equiv -1, 3, -5, 7$ (mod 16)

There will be cases when $p_1 p_2 \ldots p_k + 1$ is not $-1$ (mod 16) which is contradiction.

$p$ is even.

In this way the problem will go increasing giving same solution.

There will be primes after $E_n$.

Euclid primes are infinite.

Proved.

III. Problem 3:

In mathematics, a quasiperfect number is a theoretical natural number $n$ for which the sum of all its divisors (the divisor function $\sigma(n)$) is equal to $2n + 1$. Quasiperfect numbers are abundant numbers.

No quasiperfect numbers have been found so far, but if a quasiperfect number exists, it must be an odd square number greater than $10^{53}$ and have at least seven distinct prime factors.

Problem source: http://en.wikipedia.org/wiki/Quasiperfect_number

Solution

No odd number can be Quasiperfect number as there are even number of odd factors in odd number (except square number)

Now, all even number has at least one odd factor excluding one except the numbers of the form $2^n$.

Now, let’s say an even number has 2 odd factors: $p_1$ and $p_2$

$\Rightarrow$ There are 3 odd factors viz. $p_1$, $p_2$, $p_1 p_2$  

$\Rightarrow$ Including 1 the sum of the odd factors become even.

$\Rightarrow$ Even number who has even number of odd factors cannot be Quasiperfect number.

Now, let’s say an even number has 3 odd factors viz. $p_1$, $p_2$, $p_3$

$\Rightarrow$ There are 5 odd factors viz. $p_1$, $p_2$, $p_3$, $p_1 p_2$, $p_1 p_3$, $p_2 p_3$, $p_1 p_2 p_3$

$\Rightarrow$ Including 1 the sum of the odd factors become even.

$\Rightarrow$ Even number who has odd number of odd factors cannot be Quasiperfect number.

Now, even number of the form $2^n$ has the factor sum $1+2+2^2+\ldots+2^n=2^{n+1} - 1$. Obviously this doesn’t satisfy the condition of being a Quasiperfect number.

$\Rightarrow$ No even number can be Quasiperfect number.

Now, let’s say N is odd perfect square number.

$N = (a^n)(b^n)(c^n)(d^n)(e^n)\ldots (k^n y^n)$

Now, $1+a+a^2+\ldots+a^n= n(1+a) + 1$ (mod 4) as $a^2 \equiv 1$ (mod 4)

As a is odd $(1+a)$ is even.

$\Rightarrow$ if $(1+a)$ is not divisible by 4 otherwise it will give remainder as 1.
Problem source:

n is odd and (1+a) = 2i where i is odd.

Putting 2n+1 in place of n and 2a+1 in place of a the LHS of the equation becomes:

\[ \{1+\frac{(2n+1)}{2} + \frac{(2a+1)^2}{2} + \ldots \} + \frac{(2a+1)^4(4n+2)^4}{2} \ldots \{1+(2k+1)+(2k+1)^2+\ldots+(2k+1)^4(4y+2)^4 \} \]

Now, LHS = \{1+(2a+1)(2n+1)+1\} \ldots \{(1+k)(2y+1) + 1\} (mod 8)

\[ \text{LHS} = \{2(1+a)2(2n+1)+1\} \ldots \{2(1+k)(2y+1)+1\} \]

\[ \text{LHS} \equiv 5^{a+1} \ldots \text{mod (8) as (1+a), \ldots, (1+k) are even.} \]

\[ \text{LHS} \equiv 5 \text{ or } 1 \text{ (mod 8) if there are odd number of terms then 5, if even then 1} \]

But, RHS \equiv 2n+1 + 3 \text{ (mod 8) as any odd square number } \equiv 1 \text{ (mod 8)}

Here is the contradiction.

\[ \Rightarrow \text{Quasiperfect number doesn’t exist.} \]

Proved.

IV. Problem 4:

In mathematics, a prime triplet is a set of three prime numbers of the form \((p, p+2, p+6)\) or \((p, p+4, p+6)\). With the exceptions of \((2, 3, 5)\) and \((3, 5, 7)\), this is the closest possible grouping of three prime numbers, since every third odd number greater than 3 is divisible by 3, and hence not prime.

The first prime triplets (sequence A0098420 in OEIS) are

\((5, 7, 11), (7, 11, 13), (11, 13, 17), (13, 17, 19), (17, 19, 23), (37, 41, 43), (41, 43, 47), (67, 71, 73), (97, 101, 103), (101, 103, 107), (103, 107, 109), (107, 109, 113), (191, 193, 197), (193, 197, 199), (223, 227, 229), (227, 229, 233), (277, 281, 283), (307, 311, 313), (311, 313, 317), (347, 349, 353), (457, 461, 463), (461, 463, 467), (613, 617, 619), (641, 643, 647), (821, 823, 827), (823, 827, 829), (853, 857, 859), (857, 859, 863), (877, 881, 883), (881, 883, 887)\)

A prime triplet contains a pair of twin primes \((p \text{ and } p+2\), or \(p+4 \text{ and } p+6\)), a pair of cousin primes \((p \text{ and } p+4 \text{ or } p+2 \text{ and } p+6\)), and a pair of sexy primes \((p \text{ and } p+6)\).

A prime can be a member of up to three prime triplets - for example, 103 is a member of \((97, 101, 103), (101, 103, 107)\) and \((103, 107, 109)\). When this happens, the five involved primes form a prime quintuplet.

A prime quadruplet \((p, p+2, p+6, p+8)\) contains two overlapping prime triplets. \((p, p+2, p+6)\) and \((p+2, p+6, p+8)\).

Similarly to the twin prime conjecture, it is conjectured that there are infinitely many prime triplets. The first known gigantic prime triplet was found in 2008 by Norman Luhn and François Morain. The primes are \((p, p+2, p+6)\) with \(p = 207264482759 \times 2^{3333} - 1\). As of May 2013 the largest known prime triplet contains primes with 16737 digits and was found by Peter Kaiser. The primes are \((p, p+4, p+6)\) with \(p = 6521953289619 \times 2^{5555} - 5\).

Problem source: http://en.wikipedia.org/wiki/Prime_tripllet

\[ \text{Solution} \]

Red coloured numbers are divisible by 3. We will mark other composite numbers by blue.

Let’s say \(p_n, p_{n-2} \text{ and } p_n\) form the last triplet prime set. We assume there is finite number of prime triplet set. After \(p_n\), one of every triplet set should get divided by prime before it.

\[ p_n - 10, p_{n-8}, p_{n-6}, p_{n-4}, p_{n-2}, p_n, p_{n+2}, p_{n+4}, p_{n+6}, p_{n+8}, p_{n+10}, p_{n+12}, p_{n+14}, p_{n+16}, p_{n+18}, p_{n+20}, p_{n+22}, p_{n+24}, p_{n+26}, p_{n+28}, p_{n+30}, p_{n+32}, p_{n+34}, p_{n+36}, p_{n+38}, p_{n+40}, p_{n+42}, p_{n+44}, p_{n+46}, p_{n+48}, p_{n+50}, p_{n+52}, p_{n+54}, p_{n+56}, p_{n+58}, p_{n+60}, p_{n+62}, p_{n+64}, p_{n+66}, p_{n+68}, p_{n+70}, p_{n+72}, p_{n+74}, p_{n+76}, p_{n+78}, p_{n+80}, p_{n+82}, p_{n+84}, p_{n+86}, p_{n+88}, p_{n+90}, p_{n+92}, p_{n+94}, p_{n+96}, p_{n+98}, p_{n+100}, p_{n+102}, p_{n+104}, p_{n+106}, p_{n+108} \]

As we can see the primes occur in a regular manner if we mark the composite number as above. But primes don’t have any regular pattern to generate.

Here is the contradiction. So this case cannot happen.

\[ \Rightarrow \text{The composite number should occur in an awkward pattern} \]
More primes are necessary to make them composite as for each prime occurrence of the multiples of the primes are regular but primes don’t occur in regular pattern.

Again we form the table as below:

\[
\begin{align*}
p_n & -10 & p_n & -8 & p_n & -6 \\
p_n & -4 & p_n & -2 & p_n & \\
p_n & +2 & p_n & +4 & p_n & +6 \\
p_n & +8 & p_n & +10 & p_n & +12 \\
p_n & +14 & p_n & +16 & p_n & +18 \\
p_n & +20 & p_n & +22 & p_n & +24 \\
p_n & +26 & p_n & +28 & p_n & +30 \\
p_n & +32 & p_n & +34 & p_n & +36 \\
p_n & +38 & p_n & +40 & p_n & +42 \\
p_n & +44 & p_n & +46 & p_n & +48 \\
p_n & +50 & p_n & +52 & p_n & +54 \\
p_n & +56 & p_n & +58 & p_n & +60 \\
p_n & +62 & p_n & +64 & p_n & +66 \\
p_n & +68 & p_n & +70 & p_n & +72 \\
p_n & +74 & p_n & +76 & p_n & +78 \\
p_n & +80 & p_n & +82 & p_n & +84 \\
p_n & +86 & p_n & +88 & p_n & +90 \\
p_n & +92 & p_n & +94 & p_n & +96 \\
p_n & +98 & p_n & +100 & p_n & +102 \\
p_n & +104 & p_n & +106 & p_n & +108 \\
\end{align*}
\]

Not to form any prime triplet after \(p_n\) => there must be a composite number in each row.

Now, before \(p_n\) the number of primes is less than \(p_n\).

Now, after \(p_n\) next multiple of \(p_n\) will occur in the same column is \(p_{2n}\).

Between \(p_n\) and \(6p_n\) there are \(p_n\) number of rows.

But before \(p_n\) there are less than \(p_n\) number of primes which can make composite one number of each row.

⇒ There will be primes left in a row.

⇒ Triplet prime set is there after \(p_n\).

⇒ Triplet prime set is infinite.

Proved.

**Corollary**: Twin primes are infinite as in the case of twin prime also each row should have at least one composite number.

**V. Problem 5**

In mathematics, a superperfect number is a positive integer \(n\) that satisfies

\[
\sigma^2(n) = \sigma(\sigma(n)) = 2n,
\]

where \(\sigma\) is the divisor function. Superperfect numbers are a generalization of perfect numbers. The term was coined by Suryanarayana (1969).[1]

The first few superperfect numbers are

\[
2, 4, 16, 64, 4096, 65536, 262144 (\text{sequence A019279 in OEIS}).
\]

If \(n\) is an even superperfect number then \(n\) must be a power of 2, \(2^k\), such that \(2^{k+1}\) - 1 is a Mersenne prime.

It is not known whether there are any odd superperfect numbers. An odd superperfect number \(n\) would have to be a square number such that either \(n\) or \(\sigma(n)\) is divisible by at least three distinct primes. There are no odd superperfect numbers below \(7 \times 10^{72}\).

**Problem source**: http://en.wikipedia.org/wiki/Superperfect_number

**Solution 5**: Let \(N = (a^n)(b^m)(...)(k^y)\)

Now the equations to satisfy the condition of existing a Superperfect numbers are:

\[
(1+a+a^2+...+a^n)(1+b+b^2+...+b^m)...(1+k+k^2+...+k^y) = (p_1^{a_1})(p_2^{a_2})...(p_j^{a_j}) \text{ where } j \text{ is suffix. (A)}
\]

and

\[
(1+p_1+p_1^2+...+p_1^{a_1})(1+p_2+p_2^2+...+p_2^{a_2})...((1+p_j+p_j^2+...+p_j^{a_j}) = 2((a^n)(b^m)...(k^y)) \text{ (B)}
\]

Now if we divide equation (B) by 4 RHS \(\equiv 2 \text{ (mod 4)\)

⇒ LHS have only one odd power and rest are even otherwise LHS will be divided by 4.

Let’s say \(a_1\) is odd and rest powers are even.

Now dividing both sides of equation (A) RHS \(\equiv p_1 \text{ (mod 4)\)

⇒ RHS \(\equiv 1 \text{ (mod 4)\)

Now LHS of equation (A) should not be even.

⇒ All terms of LHS are odd.

⇒ n, m, ..., k are even

⇒ N is a square number.

Let \(N = \{a^n(b^m)(...)(k^y)\}^2 = (a^{2n})(b^{2m})...(k^{2y})\)

Now the equations to satisfy the condition of existing a Superperfect numbers are:

\[
(1+a+a^2+...+a^{2n})(1+b+b^2+...+b^{2m})...(1+k+k^2+...+k^{2y}) = (p_1^{a_1})(p_2^{a_2})...(p_j^{a_j}) \text{ where } j \text{ is suffix. (1)}
\]

and

\[
(1+p_1+p_1^2+...+p_1^{a_1})(1+p_2+p_2^2+...+p_2^{a_2})...((1+p_j+p_j^2+...+p_j^{a_j}) = 2((a^{2n})(b^{2m})...(k^{2y}))^2 \text{ (2)}
\]

Now, RHS of equation (2) \(\equiv 2 \text{ (mod 4)\) (as any odd square number is \(\equiv 1 \text{ (mod 4))\)

⇒ LHS should have only one odd power and rest are even power otherwise RHS will be divisible by 4.

Let’s say \(a_1\) is odd and rest are even.

Now RHS of equation (2) \(\equiv \{(1+p_1)(a_1+1)/2\} \{1+p_1\}(a_2/2 + 1)\)\(\ldots\)\(\{(1+p_j)(a_j/2 + 1)\)

⇒ \(1+p_1 = 2i \text{ (where } i \text{ is odd)\)

⇒ \(p_1 = 2(2i+1) - 1 = 4i+1 \text{ (putting } 2i+1 \text{ in place of } i)\)

⇒ \(p_1 \equiv 1 \text{ (mod 4)\)

Now, let’s say \(n, m, ..., k\) are odd and \((1+a), (1+b), ... (1+k)\) are divisible by 2 and not 4.

If we take them even then the problem will go on increasing giving similar results. So generalization is not violated by taking the above assumption.

Now, \(1+a+...+a^{2n} = n(1+a) + 1 \text{ (mod 4)\)

⇒ \(1+a+...+a^{2n} \equiv 3 \text{ (mod 4)\)

⇒ \(1+b+...+b^{2m} \equiv 3 \text{ (mod 4) \ldots...1+k+...+k^{2y} \equiv 3 \text{ (mod 4)\)

⇒ LHS of equation (1) \(\equiv 3 \text{ (mod 4)\)

RHS of equation (1) \(\equiv p_1 \text{ (mod 4)\) (as others are odd number square because a1 other are even)
RHS of equation (1) ≡ 1 (mod 4) (as \(p_1 \equiv 1 \mod 4\) shown above)
LHS has even number of terms otherwise LHS would be \(\equiv 3 \mod 4\)

Now, \(n(1+a) + 1 \equiv 3 \mod 4\)
\(n(1+a) = 4w+2\) form.

Here we will take \(w\) as odd. (if we take \(w\) as odd)
\(n(1+b) + 1 \equiv 7 \mod 8\)
\(m(1+b) + 1 \equiv 7 \mod 8\) ...... \(y(1+k) \equiv 7 \mod 8\)
LHS of equation (1) \(\equiv 7*7*....\) even terms \(\equiv 1 \mod 8\)

But RHS of equation (1) \(\equiv p\) (mod 8) (as other terms are odd square number \(\equiv 1 \mod 8\))
RHS of equation (1) \(\equiv 5 \mod 8\) as \(p\) is odd

Here is the contradiction.
Odd superperfect number doesn't exist.

VI. PROBLEM 6:

POLIGNAC'S CONJECTURE

In number theory, Polignac's conjecture was made by Alphonse de Polignac in 1849 and states:

For any positive even number \(n\), there are infinitely many prime gaps of size \(n\). In other words: There are infinitely many cases of two consecutive prime numbers with difference \(n\).\(^{[1]}\)

The conjecture has not yet been proven or disproven for a given value of \(n\). In 2013 an important breakthrough was made by Zhang Yitang who proved that there are infinitely many prime gaps of size \(n\) for some value of \(n < 70,000,000\).\(^{[2]}\)

For \(n = 2\), it is the twin prime conjecture. For \(n = 4\), it says there are infinitely many cousin primes \((p, p + 4)\). For \(n = 6\), it says there are infinitely many primes \((p, p + 6)\) with no prime between \(p\) and \(p + 6\).

Problem source:
http://en.wikipedia.org/wiki/Polignac%27s_conjecture

Solution

For twin prime see the corollary of Triplet Prime problem.
(End of Solution 4)

Now, we will prove there are infinitely many cousin primes. Let, the series of cousin primes is finite. Let, \(p_n - 2\) and \(p_n - 2\) form the last cousin prime pair. Red coloured numbers are divisible by 3. We will mark other composite numbers by blue.

Now we form the table as below:

\[
\begin{array}{c}
\text{p}_n + 4 \text{ p}_n + 2 \text{ p}_n + 6 \\
\text{p}_n + 8 \text{ p}_n + 10 \text{ p}_n + 12 \\
\text{p}_n + 14 \text{ p}_n + 16 \text{ p}_n + 18 \\
\text{p}_n + 20 \text{ p}_n + 22 \text{ p}_n + 24 \\
\text{p}_n + 26 \text{ p}_n + 28 \text{ p}_n + 30 \\
\end{array}
\]

The condition to satisfy that there is no cousin prime pair after \(p_n - 2\) is one of the two columns should be composite (except the column divisible by 3).

\(\Rightarrow\) The primes occur in a regular fashion.
\(\Rightarrow\) Here is the contradiction.

Now, to deny the case there needs to be composite number in the middle column in an irregular fashion.

Now, next \(p_n - 2\) occurs in the middle column after \(p_n - 2\) rows.

\(\Rightarrow\) There are more than \(p_n - 2\) composite numbers before next \(p_n - 2\) occurs in the table.
\(\Rightarrow\) But before \(p_n - 2\) there are less than \(p_n - 2\) number of primes.
\(\Rightarrow\) This case cannot happen.
\(\Rightarrow\) There are infinite number of Cousin primes.

Proved.
The conclusion can be done in other way also:
All the number of third column is composite. But there is at least difference of 2 between any consecutive prime.

\(\Rightarrow\) There will be numbers left between the series of multiples of prime before \(p_n - 2\).
\(\Rightarrow\) There will be Cousin prime generated after \(p_n - 2\).

Here is the contradiction.
\(\Rightarrow\) There are infinite number of Cousin primes.

Proved.
Now we will prove there are infinite number of sexy primes. Let, the series of sexy primes is finite. Let, \(p_n - 8\) and \(p_n - 2\) form the last sexy prime pair. Red coloured numbers are divisible by 3. We will mark other composite numbers by blue.

Now we form the table as below:

\[
\begin{array}{c}
p_n + 32 \text{ p}_n + 34 \text{ p}_n + 36 \\
p_n + 38 \text{ p}_n + 40 \text{ p}_n + 42 \\
p_n + 44 \text{ p}_n + 46 \text{ p}_n + 48 \\
p_n + 50 \text{ p}_n + 52 \text{ p}_n + 54 \\
p_n + 56 \text{ p}_n + 58 \text{ p}_n + 60 \\
p_n + 62 \text{ p}_n + 64 \text{ p}_n + 66 \\
p_n + 68 \text{ p}_n + 70 \text{ p}_n + 72 \\
p_n + 74 \text{ p}_n + 76 \text{ p}_n + 78 \\
p_n + 80 \text{ p}_n + 82 \text{ p}_n + 84 \\
p_n + 86 \text{ p}_n + 88 \text{ p}_n + 90 \\
p_n + 92 \text{ p}_n + 94 \text{ p}_n + 96 \\
p_n + 98 \text{ p}_n + 100 \text{ p}_n + 102 \\
p_n + 104 \text{ p}_n + 106 \text{ p}_n + 108 \\
\end{array}
\]
The Lonely Runner conjecture states that for each runner, there will come a time when he or she will be a distance of at least $1/k$ along the track from every other runner.

The conjecture has been proved for small values of $k (\leq 7)$.

**Solution**

Let's say, there are $i$ number of participant in the race.
There velocities are $V_1, V_2, \ldots, V_i$ where $V_2 > V_1 > \ldots > V_i$
The $j$-th person completes the cycle in $tj$ time (where $j$ is suffix)
Let’s suppose the velocities are in Arithmetic progression with common difference $d$.
Now, let’s take any participant $j$ whose velocity is $V_j$ (j is suffix)
Now $V_j$’s previous and next participants are $V(j-1)$ and $V(j+1)$ in the first cycle.
Let’s say there cannot be a lonely runner.
\[\{V(j+1) - V(j-1)\}*t_k < 2/i \] (ti is the time when first person completes the cycle)

Let’s say after time $tk$ (k is suffix) first person (with velocity $V_1$) catches person $j$ (with velocity $V_j$)
Now, $(V_1 - V_j)*tk = n$ where $n$ is number of cycle after which first person catches $j$-th person. (k is suffix)

Upto this point the $j$-th person will be between $(j+1)$-th person and $(j-1)$-th person.
This time the distance between them becomes $\{V(j+1) - V(j-1)\}*tk$ (where $j+1$, $j-1$, k are suffix)
As we have assumed no lonely runner this must be less than $2/i$
\[\{V(j+1) - V(j-1)\}*tk < 2/i \]

Putting value of $tk$ from above we get, $[[V(j+1) - V(j-1)]*n]/(V_1 - V_j) < 2/i$.
Now, $(V(j+1) - V(j-1)) = 2d$ and $(V_1 - V_j) = (j-1)*d$
Putting these values in above equation, we get,
\[2d*n/(j-1)*d < 2/i \]
\[n/(j-1) < 1/i \]
\[n < (j-1) \]
Which is impossible.
Here is the contradiction.
Let’s say the first participant meet again the $j$-th participant after $2tk$ time.
Now, the participant before and after the $j$-th participant be $m$-th and $(m+1)$-th.
\[\{V(m+1) - Vm\}2tk < 2/i \] (m+1 and m are suffix).

Proved.
Putting m = 3, 5, 7, .... and summing over all values, then equating the real and imaginary part from both sides, we get equation (1) and (2)

Now, multiplying both sides of equation (1) by \( \sin(q\ln(1/3)) \) we get,
\[
\{\cos(q\ln(1/3))\} + \{1/(3)^p\}\sin(q\ln(1/3))\cos(q\ln(3)) + \{(1/5)^p\}\sin(q\ln(1/3))\cos(q\ln(5)) + \cdots = 0
\]

......................... (3)

Now multiplying both sides of equation (2) by \( \sin(q\ln(1/3)) \) we get,
\[
\{\cos(q\ln(1/3))\} \sin(q\ln(3)) + \{(1/5)^p\}\cos(q\ln(1/3))\sin(q\ln(5)) + \{(1/5)^p\}\sin(q\ln(1/3)) + \{(1/5)^p\}\sin(q\ln(5)) + \cdots = 0
\]

......................... (4)

Now adding equation (3) and (4) we get,
\[
\{\cos(q\ln(1/3))\} + \{(1/3)^p\}\{\cos(q\ln(3)) + \cos(q\ln(5))\} + \{(1/5)^p\}\{\sin(q\ln(1/3)) + \sin(q\ln(5))\} + \{(1/5)^p\}\{\sin(q\ln(1/3)) + \sin(q\ln(5))\} + \cdots = 0
\]

\[\text{Similarly (as \( \sin(A) = \sin(A/2) \cos(A/2 + \sin(A/2) \cos(A/2) \))}
\]

\[
\{\cos(q\ln(3))\} + \{(1/5)^p\}\sin(q\ln(1/3)) + \{(1/5)^p\}\sin(q\ln(25/3)) + \{(1/5)^p\}\sin(q\ln(49/3)) + \cdots = 0
\]

\[\text{(as \( \cos(q\ln(1/3)) = \cos(q\ln(25/3)) \) because \( \cos(-A) = \cos(A) \))}
\]

\[
\{\cos(q\ln(3))\} = 0
\]

\[
\cos((2n\pi + \pi/2)) = 0
\]

\[\text{(where n is integer)}
\]

\[
q = (2n + 1)\pi/\ln(3)
\]

Similarly it can be proved for any odd integer 2a+1, q = (2n+1)\pi/ln(2a+1) where a = 1, 2, 3, ...

Problem 10:

In 1967, L. J. Lander, T. R. Parkin, and John Selfridge (LPS) conjectured that if
\[
\sum_{i=1}^{n} a_i^k = \sum_{j=1}^{m} b_j^k
\]

where \( a_i \neq b_j \) are positive integers for all \( 1 \leq i \leq n \) and \( 1 \leq j \leq m \), then \( m+n \geq k \)

The problem is to prove the conjecture or find a counter-example
Problem source: http://en.wikipedia.org/wiki/Lander,_Parkin_and_Selfridge_conjecture

Solution

Let, m+n < k

We will prove this with assumption m < n < 2m because similarly can be proved for md < n < m(d+1) where d = [n/m] (where [x] = greatest integer contained in x) Let the ordered set of a₁, a₂, ..., an (n is suffix) = p₁, p₂, ..., pn (n is suffix) where p₁ < p₂ < ... < pn Similarly, ordered set of b₁, b₂, ..., bm (m is suffix) = q₁, q₂, ..., qm where q₁ < q₂ < ... < qm

Now applying AM ≥ GM on p₁ᵏ, p₂ᵏ, ..., pnᵏ we get,

\[
\frac{(p₁ᵏ) + (p₂ᵏ) + ... + (pnᵏ)}{n} ≥ (p₁p₂...pn)^(k/n)
\]

Now multiplying both the equations, we get,

\[
\frac{(q₁ᵏ) + (q₂ᵏ) + ... + (qmᵏ)}{m} ≥ (q₁q₂...qm)^(k/m)
\]

Now, pn * q₁/qm > 1, p(n-1) * q₂/qm > 1, ....
p(n-m-1) * qm/qm > 1 (this is trivial)

Now, qm > qi (where i, m are suffix and i = 1, 2, ..., m-1)

\[
1 > qi/qm
\]

\[
1 > (qi/qm)^(k-1)
\]

\[
qi/qm > (qi/qm)'^k [Multiplying both sides by qi/qm]
\]

\[
(pn*qi/qm > (q/qm)^k, (p(n-1)*q₂/qm > (q/qm)^k, ...., (p(n-m-2)*q(m-1)/qm > (q(m-1)/qm)^k
\]

Equation (A) cannot hold true.

Our assumption was wrong.

m+n ≥ k

The conjecture is true.

REFERENCES


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