Rational Distance, 4-D Euler Brick, ABC and Goldbach Conjecture

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Abstract— **In this paper we will prove two problems :**

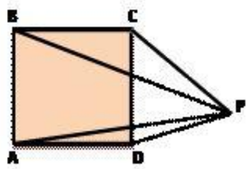
1) **Rational Distance problem**
2) **4-D Euler Brick.**
3) **ABC Conjeture**
4) **Goldbach Conjecture**

First we will show no such point exist whose distances from every corner of a integer sided square are integers.
Then we will prove non-existence of 4-D Euler Brick.
Then we will prove ABC Conjecture is true.
Then we will prove Goldbach Conjecture is true.

Index Terms— **Conjecture, 4-D Euler Brick, Pythagorean triple.**

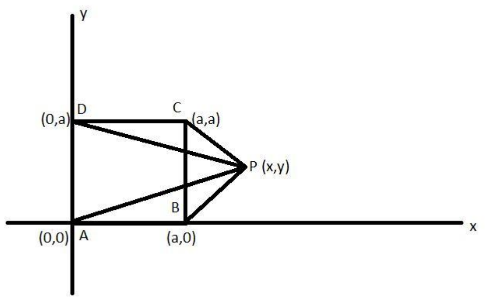
I. Rational Distance Problem

Given a unit square, can you find any point in the same plane, either inside or outside the square, that is a rational distance from all four corners? Or, put another way, given a square ABCD of any size, can you find a point P in the same plane such that the distances AB, PA, PB, PC, and PD are all integers?



The problem is to find such a point, or prove that no such point can exist.

A. Solution :



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Let ABCD be the square. The co-ordinates are shown in the image. P be any point on the same plane. a is an integer.

$PA^2 = x^2 + y^2$

$PB^2 = (x-a)^2 + y^2 = x^2 + y^2 + a^2 - 2ax = PA^2 + a^2 - 2ax$

⇨ $2ax = PA^2 + a^2 - PB^2$

$PD^2 = x^2 + (y-a)^2 = x^2 + y^2 + a^2 - 2ay = PA^2 + a^2 - 2ay$

⇨ $2ay = PA^2 + a^2 - PD^2$

$PC^2 = (x-a)^2 + (y-a)^2 = x^2 + y^2 + 2a^2 - 2ax - 2ay$

⇨ $PC^2 = PA^2 + 2a^2 - (PA^2 + a^2 - PB^2) – (PA^2 + a^2 - PD^2)$

⇨ $PC^2 + PA^2 = PB^2 + PD^2$

Let's say PA, PB, PC, PD all cannot be even. Otherwise the equation will get divided by 4 until one becomes odd.
Let's say PC, PA even and PB, PD odd
PC, PA even => LHS divisible by 4.
PB, PD odd => RHS ≡ 2 (mod 4) (as square of any odd integer ≡ 1 (mod 4))
Contradiction.
Let's say PC and PB even and PA and PD odd.
Now, $PC^2 + PA^2 = PB^2 + PD^2$

⇨ $PC^2 - PB^2 = PD^2 - PA^2$

Putting $PC = 2m_1$, $PB = 2m_2$, $PD = 2m_3+1$ and $PA = 2m_4 + 1$

$4(m_1^2 - m_2^2) = 4m_3^2 + 4m_3 + 1 – 4m_4^2 - 4m_4 - 1$

⇨ $m_1^2 - m_2^2 = (m_3 + m_4 + 1)(m_3 - m_4)$

Let's say m_3, m_4 both odd. Putting $(2m_3+1)$ in place of m_3 and $(2m_4+1)$ in place of m_4
The equation becomes, $m_1^2 - m_2^2 = (2m_3 + 1 + 2m_4 + 1 + 1)(2m_3 + 1 – 2m_4 - 1)$

⇨ $m_1^2 - m_2^2 = 2(2m_3+2m_4+3)(m_3 - m_4)$

In LHS both are square. So, multiple can be minimum 4 and not 2.
So, this equation is impossible.
Let's say m_3 even and m_4 odd. Putting $2m_3$ in place of m_3 and $2m_4+1$ in place of m_4
The equation becomes, $m_1^2 - m_2^2 = (2m_3 + 2m_4 +1 +1)(2m_3 - 2m_4 -1)$

⇨ $m_1^2 - m_2^2 = 2(m_3+m_4+1)(2m_3-2m_4-1)$

Again LHS can have at least multiple of 4 so this equation is not possible.
Let's say m_3, m_4 both even. Putting $2m_3$ in place of m_3 and $2m_4$ in place of m_4.
The equation becomes, $m_1^2 - m_2^2 = (2m_3+2m_4+1)(2m_3-2m_4)$

⇨ $m_1^2 - m_2^2 = 2(2m_3+2m_4+1)(m_3 - m_4)$

Again LHS can have at least multiple of 4 so this equation is not possible.
Np other combination of m_3 and m_4 possible.

⇨ The equation has no solution.

⇨ The point P doesn't exist.

B. Synopsis :

Euler bricks are bricks with all edges and face diagonals integers, named after one of their first investigators, Leonhard Euler (1707-1783). Each pair of edges forms the legs of a Pythagorean triple, each face diagonal the hypotenuse of a Pythagorean triple. Example edges: 88, 234, 480.

The term Euler brick has a number of aliases - uses the term Classical Rational Cuboid. uses the term Body Cuboid to refer to an Euler brick with an irrational body diagonal. French terms include brique de Pythagore and paralleloide de Pythagore. Currently it is not known if an Euler brick can have an integer body diagonal - this question also has a number of aliases - the Perfect Box Problem, Perfect Cuboid Problem, Rational Box Problem, Rational Cuboid Problem, Integer Cuboid Problem, etc.

II. Euler Brick

An Euler brick is a cuboid that possesses integer edges $a > b > c$ and face diagonals

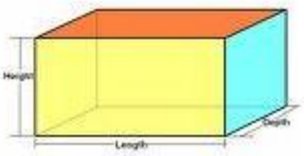
$$d_{ab} = \sqrt{a^2 + b^2}$$
$$d_{ac} = \sqrt{a^2 + c^2}$$
$$d_{bc} = \sqrt{b^2 + c^2}.$$

If the space diagonal is also an integer, the Euler brick is called a perfect cuboid, although no examples of perfect cuboids are currently known.

The smallest Euler brick has sides $(a, b, c) = (240, 117, 44)$ and face polyhedron diagonals $d_{ab} = 267$, $d_{ac} = 244$, and $d_{bc} = 125$, and was discovered by Halcke (1719; Dickson 2005, pp. 497-500). Kraitchik gave 257 cuboids with the odd edge less than 1 million (Guy 1994, p. 174). F. Helenius has compiled a list of the 5003 smallest (measured by the longest edge) Euler bricks. The first few are (240, 117, 44), (275, 252, 240), (693, 480, 140), (720, 132, 85), (792, 231, 160)

III. 4-D Euler Brick Problem

An Euler Brick is just a cuboid, or a rectangular box, in which all of the edges (length, depth, and height) have integer dimensions; and in which the diagonals on all three sides are also integers.



So if the length, depth and height are a, b, and c respectively, then a, b, and c are integers, as are the quantities $\sqrt{(a^2+b^2)}$ and $\sqrt{(b^2+c^2)}$ and $\sqrt{(c^2+a^2)}$.

The problem is to find a four dimensional Euler Brick, in which the four sides a, b, c, and d are integers, as are the six face diagonals $\sqrt{(a^2+b^2)}$ and $\sqrt{(a^2+c^2)}$ and $\sqrt{(a^2+d^2)}$ and $\sqrt{(b^2+c^2)}$ and $\sqrt{(b^2+d^2)}$ and $\sqrt{(c^2+d^2)}$, or prove that such a cuboid cannot exist .

A. Solution

Let a, b, c, d are the sides of the four dimensional Euler Brick.
Now the equations are :

$a^2 + b^2 = e^2$ ....(1)
$b^2 + c^2 = f^2$............(2)
$c^2 + a^2 = g^2$.............(3)
$b^2 + d^2 = h^2$............(4)
$c^2 + d^2 = i^2$..............(5)
$a^2 + d^2 = j^2$..............(6)

Case 1 : 3 of a, b, c, d even and one odd.

Let's say a, b, c even and d is odd.
From (1) if we divide the equation by 4 then, a and e say becomes odd.

⇨ b^2 has a factor of 16

Now if we divide equation (3) by 4 let's say a and g become odd.

⇨ c^2 has a factor of 16.

(1) Now if we divide the equation (2) by 16 then in left side b^2 and
(2) c^2 becomes odd f^2 should remain even. => f^2 has got a factor of 64.
(3) Now $b = 4(2p+1)$ and $c = 4(2q+1)$ (as they have got factor of 16)

⇨ $b^2 = 16(4p^2 + 4p +1)$ and $c^2 = 16(4q^2 + 4q + 1)$
⇨ $b^2 = 64p(p+1) + 16$ and $c^2 = 64q(q+1) + 16$
⇨ $b^2 + c^2 \equiv 16+16 =32 \pmod{64}$

But $f^2 \equiv 0 \pmod{64}$
Equation (2) doesn't hold.
Contradiction.
Now if, b^2 or c^2 have a factor of 64 then also the equation fails to exist.

⇨ b^2 or c^2 must have a factor of 64*4 = 256.

Let's say c^2 has a factor of 256.

⇨ b^2 has a factor of 64.

From equation (5), $c^2 + d^2 = i^2$

⇨ i^2 of the form $256m_1 + p_1$

From equation (6), $a^2 + d^2 = j^2$

⇨ j^2 is of the form $16m_2 + p_2$

From equation (3), $c^2 + a^2 = g^2$

⇨ g^2 is of the form $16(16m_3+p_3)$

Now subtracting equation (5) – (6) we get,
$c^2 - a^2 = i^2 - j^2$
$c^2+ a^2 = g^2$ (from equation (3))
Now, solving for c^2 we get, $c^2 = (i^2 - j^2 + g^2)/2$
Now putting the values of i^2 , j^2, g^2 we get,
$c^2 = (256m_1 + p_1 - 16m_2 – p_2 + 16(16m_3 + p_3))/2$
Let's consider the worst possible scenario. Let's $p_1 – p_2 = 64m_4$
$c^2 = 16(16m_1 + 4m_4 -m_2 + 16m_3 + p_3)/2$
Let's put $2m_2 + 1$ in place of m_2 . p_3 = 1 or 9

$c^2 = 16(16m_1 + 4m_4 - 2m_2 -1 +16m_3 + 1) /2$

- ⇨ $c^2 = 32(8m_1 + 2m_4 - m_2 +8m_3)/2$ which clearly cannot hold.

So, no solution for 3 even 1 odd.

Case 2 : At least 2 is odd. Let's say a, b are odd and c, d are even.

From equation (1) we get e^2 = even

- ⇨ e^2 has a factor of 4.

Now a, b $\equiv \pm 1 \pmod 4$

- ⇨ $a^2 , b^2 \equiv 1 \pmod 4$
- ⇨ $a^2 + b^2 \equiv 2 \pmod 4$

But $e^2 \equiv 0 \pmod 4$

Here is the contradiction.

B. Conclusion :

From Case 1 and Case 2 we see that no other combination of a, b, c, d exists to give solution to the 6 simultaneous equations.

C. Result :

4D Euler Brick doesn't exist.

IV. ABC Conjecture Problem

Let A, B, and C be three coprime integers such that

A + B = C

Now multiply together all the distinct primes that divide any of these numbers, and call the result rad(ABC).

For example, if we start with 4 + 11 = 15, we have 2 (which divides 4), 11 (which divides 11) and 3 and 5 (which divide 15), so rad(ABC) = 2 x 11 x 3 x 5 = 330.

C is almost always smaller than rad(ABC), but not always. If you start with 2 + 243 = 245, the primes are 2 (which divides 2), 3 (which divides 243), and 5 and 7 (which divide 245). So rad(ABC) = 2 x 3 x 5 x 7 = 210. In this case, C is much bigger than rad(ABC).

Let's count C as "much bigger" whenever it's bigger than $\text{rad}(ABC)^{1.1}$ or $\text{rad}(ABC)^{1.001}$ or $\text{rad}(ABC)^{1.000000000001}$ or $\text{rad}(ABC)^{1+\epsilon}$. The ABC conjecture says that no matter how small ϵ, there will still be only finitely many examples where C counts as much bigger than rad(ABC).

The problem is to prove or disprove the conjecture.

A. Solution :

A + B = C

Let {P} be the set of primes.

Let's say C takes prime values only. So C ϵ {P}

Let us take any prime number say 13.

2 + 11 = 13

Here 2, 11, and 13 are co-prime and rad(A)rab(B)rad(c) = 2*11*13 > C =13

So, if we take C as prime then rad(C) = C

- ⇨ Rad(A)rad(B)rad(C) > C

As prime number series is infinite. For every prime we will find a C.

- ⇨ Rad(A)rad(B)rad(C) > C is true for infinite values of C.

Now we will consider the case when rad(A)rad(B)rad(c) < C

Say, there are infinite C's for which this equation hold.

Let this set call Q where Q ={C when rad(A)rad(B)rad(c) < C}

As {P} the set of primes is infinite so there must be one to one or one to many relation between {P} and {Q} for each element.

Say for C = 11 all combinations of A and B, C< rad(A)rad(B)rad(C)

For 11 we cannot find any value of C in {Q} for which that gets related to one or many elements of P by a relation.

Now, the set {Q}'s all elements are not related to {P}

- ⇨ Q is not an infinite set.

Now, Q can be empty.

Now, $2 + 3^5 = 5*7^2$

Here rad(A) = 2, rad(B) = 3 , rad(C) = 5*7

Now, rad(A)rad(B)rad(C) = 2*3*5*7 = 210 and C = 245

- ⇨ Rad(A)rad(B)rad(C) < C
- ⇨ Q is non-empty.

So, Q is a finite non-empty set.

- ⇨ There are few (finite number of C) C's for which rad(A)rad(B)rad(C) < C
- ⇨ ABC conjecture is true.

Peoved.

V. Goldbach Conjecture Problem

The Goldbach Conjecture states that every even number greater than 2 is the sum of two primes. A number is prime if it is divisible only by itself and 1. So, for example, 36 = 17+19.

The problem is to prove the conjecture, or find a counter-example.

A. Solution :

$p + p_1 = 2k$ ....(1)

where p, p_1 are primes and k ϵ N (Natural number set)

Now, the set of prime number is countable infinite. Let's call it {P}

N is also infinite.

There is one to one or one to many relationship between Countable infinite sets.

Implies, we will find at least one p and p_1 for every natural number k in equation (1)

Implies Goldbach Conjecture is true.

Corollary : Every even integer greater than 2 is sum of two ***distinct*** primes. The representation is ***at least*** one.

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