Odd Perfect Number

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Abstract— **A perfect number is a positive integer that is equal to the sum of its positive divisors, and can be represented by the equation σ(n) = 2n. Even perfect numbers have been discovered, and there is a search that continues for odd perfect number(s). A list of conditions for odd perfect numbers to exist has been compiled, and there has never been a proof against their existence.**

Index Terms— **positive divisors, Odd Perfect Number**

I. Introduction

A number N is perfect if the sum of its divisors, including 1 but excluding itself, add up to N.

So, for example, 28 is perfect because 1 + 2 + 4 + 7 + 14 = 28.

The problem is to find an odd perfect number, or prove that no such number exists.

II. Solution

Let's say the number is (a^m)(b^n)(c^p)……(k^y) (where a, b, …p are odd primes)

Now, sum of all factors of the number including itself is : (1+a+a^2+a^3+....+a^n)(1+b+b^2+......+b^m)(1+c+c^2+c^3 +…..+c^p)…..(1+k+k^2+k^3+….k^y)

This should be equal to twice the number if odd perfect number exists.

The equation is
(1+a+a^2+a^3+....+a^n)(1+b+b^2+......+b^m)(1+c+c^2+c^3 +…..+c^p)…..(1+k+k^2+k^3+….k^y) = 2(a^m)(b^n)(c^p)……(k^y) ....(A)

To hold the equality :

1) The number that is ≡ 3(i.e. -1)(mod 4) cannot have odd power. because otherwise the left hand side will be divisible by 4 but right side is divisible by 2.

2) The number that is ≡1(mod 4) cannot have power of the form (2^n-1). Because otherwise the left hand side will be divisible by 4 whereas right side is divisible by 2.

3) The number cannot have more than one odd power. otherwise the left hand side will be divisible by 4 whereas right side is divisible by 2.

4) Only one number can have odd power. If two numbers have odd power then left hand side will be divided by 4 whereas right hand side only by 2.

Now, let's say a has the odd power. Rest of the powers are even.

We will prove it for 2 primes.
Let the number be N = (a^n)*(b^m)
The equation to satisfy the condition of perfect number is,
(1+a+a²+.....+a^n)(1+b+b²+....+b^m) = 2*(a^n)(b^m)

⇨ (1+a)(1+a²+....+a^(n-1))(1+b+b²+.....+b^m) = 2*(a^n)(b^m) ...... (A)

Let's say (1+a) = 2*(b^k)
Putting the value of (1+a) in equation (A) we get,
(1+a²+....+a^(n-1))(1+b+b²+......+b^m) = (a^n)*b^(m-k)
Now, say, without loss of generality, (1+b+b²+....+b^m) = $(a^{k_1})(b^{k_2})$
If we divide this equation by b then LHS ≡1 and RHS ≡ 0 (mod b)

⇨ $k_2 = 0$.

So, the equation becomes, (1+b+b²+....+b^m) = a^{k_1} ....(B)
Putting value from equation (B) into equation (A) we get,
1+a²+.....+a^(n-1) = {$a^{(n-k_1)}$}{b^(m-k)}
Now, dividing both sides of this equation by a, we get,
LHS ≡ 1 and RHS ≡0

⇨ $n-k_1 = 0$

⇨ $k_1 = n$

Putting $k_1 = n$ in equation (B) we get,
(1+b+b²+.....b^m) = a^n ....(C)
Now, a ≡ -1 (mod b) (as a+1 = 2b^k)

⇨ a^n ≡ -1 (mod b) (as n is odd)

Now dividing both sides of the equation (C), we get,
LHS ≡ 1 and RHS ≡ -1
Contradiction.

So, for N = (a^n)(b^m) , N cannot be a perfect number.

Now we will prove for 3 primes.
Let's say N = (a^n)(b^m)(c^p)
Where n is odd and m, p are even.
To satisfy the condition of perfect number, the equation is ,

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$(1+a+a^2+.....+a^n)(1+b+b^2+.....+b^m)(1+c+c^2+....+c^p) = 2*(a^n)(b^m)(c^p)$

⇨ $(1+a)(1+a^2+....+a^(n-1))$
$(1+b+b^2+.....+b^m)(1+c+c^2+....+c^p) = 2*(a^n)(b^m)(c^p)$

Let's say $1+a = 2*(b^{k_1})(c^{k_2})$
The equation becomes,
$(1+a)(1+a^2+....+a^(n-1))$
$(1+b+b^2+.....+b^m)(1+c+c^2+....+c^p) =$
$(a^n)\{b^(m-k_1)\}\{c^(p-k_2)\}$ .....(A)
Now, Let's say $(1+a^2+....a^(n-1)) = (b^{k_3})(c^{k_4})$ (a cannot be a factor otherwise dividing by a will give contradictory result.) ....(1)
Now, $a^2 \equiv 1$ (mod b or c)
Now, dividing equation (1) by b and c respectively , we get,
$(n+1)/2 = bc(a^{k_9})$ (RHS ≡ 0 and LHS ≡ (n+1)/2 in both cases mod(b) and mod(c)) ......(2)
Now, dividing equation (A) by b and c respectively, we get,
$\{(n+1)/2\}(1+c+c^2+....+c^p) = b(c^{k_5})(a^{k_6})$ (as RHS ≡ 0 and LHS will become some multiple of b).....(3)
$((n+1)/2\}(1+b+b^2+.....+b^m) = c(b^{k_7})(a^{k_8})$ ....(4)
Now multiplying both the above equations and putting value in equation (A) we get,
$(1+a^2+.....+a^(n-1)) =$
$\{(n+1)/2\}^2.[a^\{n-(k_6+k_8)\}]\{b^(m-k_1-k_7-1)\}\{c^(p-k_2-k_5-1)\}$
Dividing both sides by a we get LHS ≡ 1 (mod a) & RHS ≡ 0.

⇨ $n = k_6 + k_8$

Now, putting value of (2) in (3) and (4) we get,
$(1+c+c^2+.....+c^p) = \{c^(k_5-1)\}\{a^(k_6 - k_9)\}$ ....(5)
$(1+b+b^2+.....+b^m) = \{b^(k_7-1)\}\{a^(k_8-k_9)\}$.......(6)
Now dividing both sides of equation (5) by c we get RHS ≡ 0 and LHS ≡ 1 => $k_5 = 1$
Similarly dividing both sides of equation (6) by b we get, k_7= 1.
Now equation (5) is : $(1+c+....+c^p) = a^(k_6 - k_9)$
Now equation (6) is : $(1+b+....+b^m) = a^(k_8-k_9)$
Putting this values in equation (A) we get, $k_9 = 0$ (as $k_6+k_8 = n$)
Now equation(5) is : $(1+c+....+c^p) = a^{k_6}$ ....(7)
Now equation (6) is : $(1+b+....+b^m) = a^{k_8}$.........(8)
Now, $k_6 + k_8 = n$ = odd.

⇨ One of k_6 and k_8 is odd and another one is even.

Let's say k_6 is odd.
Now dividing both sides of equation (7) by c we get LHS ≡ 1 (mod c) and RHS ≡ -1 (mod c) [as a ≡ -1 (mod c) => a^{k_6} ≡ -1 (mod c) as k_6 is odd]
Here is the contradiction.

Now when, $a+1 = 2(b^k)$
Equation (A) becomes,
$(1+a^2+....+a^(n-1))(1+b+....+b^m)(1+c+...+c^p) =$
$(a^n)\{b^(m-k)\}(c^p)$
Dividing both sides by b,
$\{(n+1)/2\}(1+c+...+c^p) = b(c^{k_1})(a^{k_2})$
Now dividing this equation by c,
$(n+1)/2 = c(b^{k_3})(a^{k_4})$
Putting the value of (n+1)/2 in above equation we get,
$\{b^(k_3-1)\}(1+c+...+c^p) = \{c^(k_1-1)\}\{a^(k_2-k_4)\}$

⇨ $k_3 = 1$(as a, b, c are primes)

⇨ $(1+c+....+c^p) = \{ c^(k_1-1)\}\{a^(k_2-k_4)\}$

⇨ $k_1 = 1$ (On dividing both sides by c)

⇨ $(1+c+...+c^p) = a^(k_2-k_4)$ ....(5)

Now, putting this value on equation (A),
$(1+a^2+.....+a^(n-1))(1+b+....+b^m) =$
$\{a^(n-k_2+k_4)\}\{b^(m-k)\}(c^p)$ .....(B)
Dividing both sides by a , we get,
$1+b+....+b^m = a(b^{k_5})(c^{k_6})$ ....(1)

⇨ $k_5 = 0$ (On dividing both sides by b)

⇨ $1+b+....+b^m = a(c^{k_6})$ ....(4)

Now, from (B), $1+a^2+...+a^(n-1) =$
$\{a^(n-k_2+k_4-1)\}\{b^(m-k)\}\{c^(p-k_6)\}$

⇨ $n-1 = k_2-k_4$(Dividing both sides by a)

⇨ $1+a^2+....+a^(n-1) = \{b^(m-k)\}\{c^(p-k_6)\}$

Dividing both sides of equation (4) by c we get,
$1+b+...+b^m = c(b^{k_7})(a^{k_8})$

⇨ $k_7 = 0$ (On dividing both sides by b)

⇨ $1+b+...+b^m = c(a^{k_8})$ ......(2)

Now, equating RHS of equation (1) and (2) we get,
$a(c^{k_6}) = c(a^{k_8})$

⇨ $a^(k_8-1) = c^(k_6-1)$

This equation has solution only when $k_8 =1$ & $k_6 = 1$ (as a, c both prime)
Therefore, $1+b+.....+b^m = ac$ .... (C)
Dividing both sides by b we get, c ≡ -1 (mod b)
Now $c = 2j(b^i) -1$ (say)
Now $ac = (2b^k – 1)(2jb^i – 1) = 4jb^(k+i) – 2b^k – 2jb^i + 1$
Putting this in equation (C), we get,
$1+b+....+b^m = 4jb^(k+i) – 2b^k – 2jb^i +1$

⇨ $b+b^2+....+b^m = 4jb^(k+i) – 2b^k – 2jb^i$

⇨ $1+b+...+b^(m-1) = 4jb^(k+i-1) – 2b^(k-1) – 2jb^(i-1)$

Dividing both sides by b, LHS ≡ 1 and RHS ≡ 0.
Here is the contradiction.
So, when $N = (a^n)(b^m)(c^p)$ then N is not a perfect number.

Now, we will prove for 4 primes raised to powers.

Let's say $N = (a^n)(b^m)(c^p)(d^q)$

To hold the equality for perfect number,

$(1+a+a^2+....+a^n)(1+b+....+b^m)(1+c+....+c^p)(1+d+...+d^q) = 2*(a^m)(b^n)(c^p)(d^q)$

⇨ $(1+a)(1+a^2+...+a^(n-1))$
$)(1+b+....+b^m)(1+c+....+c^p)(1+d+...+d^q) =$
$2*(a^m)(b^n)(c^p)(d^q)$

Now, Let's say $1+a = 2*(b^{k_1})(c^{k_2})(d^{k_3})$

Putting the value of (1+a) the equation becomes,

$(1+a^2+...+a^{(n-1)}))(1+b+....+b^m)(1+c+....+c^p)(1+d+...+d^q) = 2*(a^n)\{b^{(m-k_1)}\}\{c^{(p-k_2)}\}\{d^{(q-k_3)}\}$ ....(A)

Now without loss of generality we can write,

$(1+a^2+.....+a^{(n-1)}) = (b^{k_4})(c^{k_5})(d^{k_6})$ (as a cannot be a multiple in right side)

Dividing both sides by b, c, d respectively we find RHS ≡ 0 and LHS ≡ (n+1)/2

So, we can write, $(n+1)/2 = bcd*a^{k_7}$ ....(1)

Dividing both sides of equation (A) by b we get,

RHS ≡ 0 and LHS ≡ $\{(n+1)/2\}(1+c+....+c^p)(1+d+.....+d^q)$

We can write, $\{(n+1)/2\}(1+c+....+c^p)(1+d+.....+d^q) = b*(c^{k_8})(d^{k_9})(a^{k_{10}})$ .....(2)

Now dividing this equation by c we get, RHS ≡ 0 and LHS should be multiple of c

⇨ $\{(n+1)/2\}(1+d+....+d^q) = c*(b^{k_{11}})(d^{k_{12}})(a^{k_{13}})$

Putting value of (n+1)/2 from equation (1) we get,

$1+d+...+d^q = \{b^{(k_{11}-1)}\}\{d^{(k_{12}-1)}\{a^{(k_{13}-k_7)}\}$

Now dividing both sides by d gives LHS ≡ 1 & RHS ≡ 0 => $k_{12} = 1$

Equation becomes, $1+d+....+d^q = \{b^{(k_{11}-1}\}\{a^{(k_{13}-k_7)}\}$ ....(3)

Dividing equation (2) by d we get, RHS ≡ 0 and LHS ≡ $\{(n+1)/2\}(1+c+....+c^p)$

So, we can write, $\{(n+1)/2\}(1+c+....+c^p) = d*(b^{k_{14}})(c^{k_{15}})(a^{k_{16}})$

Putting value of (n+1)/2 from equation (1) we get,

$1+c+.....+c^p = \{b^{(k_{14}-1)}\}\{c^{(k_{15}-1)}\}\{a^{(k_{16}-k_7)}\}$

Dividing both sides by c gives LHS ≡ 1 & RHS ≡ 0 => $k_{15} = 1$

Equation becomes, $1+c+.....+c^p = \{b^{(k_{14}-1)}\}\{a^{(k_{16}-k_7)}\}$ ....(4)

Dividing equation (A) by c we get,

$\{(n+1)/2\}(1+b+....+b^m)(1+d+...+d^q) = c*(b^{k_{17}})(d^{k_{18}})(a^{k_{19}})$ .....(5)

Dividing both side of equation (5) by d we get,

$\{(n+1)/2\}(1+b+....+b^m) = d*(b^{k_{20}})(c^{k_{21}})(a^{k_{22}})$

Putting value of (n+1)/2 from equation (1) we get,

$1+b+....+b^m = \{b^{(k_{20}-1)}\}\{c^{(k_{21}-1)}\}\{a^{(k_{22}-k_7)}\}$

Now dividing both side by b we get RHS ≡ 0 but LHS ≡ 1 => $k_{20} = 1$

Equation becomes, $1+b+....+b^m = \{c^{(k_{21}-1)}\}\{a^{(k_{22}-k_7)}\}$........... (6)

Now dividing both sides of equation (5) by b we get,

$\{(n+1)/2\}(1+d+....+d^q) = b*(d^{k_{24}})(a^{k_{25}})(c^{k_{26}})$

Putting value of (n+1)/2 from equation (1) we get,

$1+d+.....+d^q = \{c^{(k_{26}-1)}\}\{d^{(k_{24}-1)}\}\{a^{(k_{25}-k_7)}\}$

Now dividing both sides by d we get RHS ≡ 0 whereas LHS ≡ 1 => $k_{24} = 1$

Now the equation becomes, $1+d+.....+d^q = \{c^{(k_{26}-1)}\}\{a^{(k_{25}-k_7)}\}$ ...... (7)

Now equating RHS of equation (3) and (7), we get

$\{b^{(k_{11}-1}\}\{a^{(k_{13}-k_7)}\} = \{c^{(k_{26}-1)}\}\{a^{(k_{25}-k_7)}\}$

⇨ $b^{(k_{11}-1)} = \{c^{(k_{26}-1)}\}\{a^{(k_{25}-k_{13})}\}$

As, a, b, c are all prime numbers this equation can only hold when,

$k_{11} = 1$, $k_{26} = 1$ and $k_{25} = k_{13}$

Therefore, Equation (3) or (7) becomes,

$1+d+.....+d^q = a^{(k_{25}-k_7)}$ ....(13)

Now, Dividing equation (A) by d we get,

$\{(n+1)/2\}(1+b+.....+b^m)(1+c+....c^p) = d*(b^{k_{23}})(c^{k_{24}})(a^{k_{25}})$ .....(8)

Now dividing both side by c we get,

$\{(n+1)/2\}(1+b+.....+b^m) = c*(b^{k_{27}})(d^{k_{28}})(a^{k_{29}})$

Putting value of (n+1)/2 from equation (1) we get,

$1+b+.....+b^m = \{b^{(k_{27}-1)}\}\{d^{(k_{28}-1)}\}\{a^{(k_{29}-k_7)}\}$

Dividing both sides by b RHS ≡ 0 whereas LHS ≡ 1 giving $k_{27} = 1$.

Equation becomes, $1+b+.....+b^m = \{d^{(k_{28}-1)}\}\{a^{(k_{29}-k_7)}\}$..........(9)

Now equating RHS of equation (6) and (9) we get,

$\{c^{(k_{21}-1)}\}\{a^{(k_{22}-k_7)}\} = \{d^{(k_{28}-1)}\}\{a^{(k_{29}-k_7)}\}$

⇨ $c^{(k_{21}-1)} = = \{d^{(k_{28}-1)}\}\{a^{(k_{29}-k_{22})}\}$

As c, d, a are prime this equation can only hold when,

$k_{21} = 1$, $k_{28} = 1$ and $k_{29} = k_{22}$

So, (6) or (9) becomes, $1+b+...+b^m = a^{(k_{29}-k_7)}$ .... (10)

Now, dividing equation (8) by b we get,

$\{(n+1)/2\}(1+c+....+c^p) = b*(c^{k_{30}})(d^{k_{31}})(a^{k_{32}})$

Putting value of (n+1)/2 from equation (1) we get,

$1+c+....+c^p = \{c^{(k_{30}-1)}\}\{d^{(k_{31}-1)}\}\{a^{(k_{32}-k_7)}\}$

Dividing both sides by c gives RHS ≡ 0 whereas LHS ≡ 1 giving $k_{30} = 1$

Equation becomes, $1+c+...+c^p = \{d^{(k_{31}-1)}\}\{a^{(k_{32}-k_7)}\}$ ....(11)

Now, equating RHS of equation (4) and (11) we get,

$\{b^{(k_{14}-1)}\}\{a^{(k_{16}-k_7)}\} = \{d^{(k_{31}-1)}\}\{a^{(k_{32}-k_7)}\}$

⇨ $b^{(k_{14}-1)} = \{d^{(k_{31}-1)}\}\{a^{(k_{32}-k_{16})}\}$

Now, as b, d, a all are primes, this equation can only hold when,

$k_{14} = 1$, $k_{31} = 1$ and $k_{32} = k_{16}$

Equation, (4) or (11) becomes, $1+c+...+c^p = a^{(k_{32}-k_7)}$ ....(12)

Now, let's say $k_{32}-k_7 = k_{33}$, $k_{29}-k_7 = k_{34}$, $k_{25}-k_7 = k_{35}$

Equation (10) => $1+b+...+b^m = a^{k_{34}}$ ....(14)

Equation (12) => $1+c+...+c^p = a^{k_{33}}$ ....(15)

Equation (13) => $1+d+.....+d^q = a^{k_{35}}$ ....(16)

Putting these values in equation (A) we get,

$(1+a^2+....+a^{(n-1)}) = \{a^{(n-k_{33}-k_{34}-k_{35})}\}\{b^{(m-k_1)}\}\{c^{(p-k_2)}\}\{d^{(q-k_3)}\}$

Dividing both side by a, we get, $n = k_{33}+k_{34}+k_{35}$ = odd

⇨ One of k_{33}, k_{34}, k_{35} must be odd. (because the combination can be (odd+even+even) or (odd+odd+odd))

Let's say k_{33} is odd.

Now, dividing equation (15) by c LHS ≡ 1 and RHS ≡ -1 (as a≡-1(mod c))

Here is the contradiction.

Similarly we can prove when $(1+a) = 2*(b^{k_1})(c^{k_2})$ or $2*(b^{k_1})$ as we have proved for $N= (a^n)(b^m)(c^p)$.

So, when $N=(a^n)(b^m)(c^p)(d^q)$ then also N is not perfect number.

In this way we can also prove for $N = (a^n)(b^m).....(k^y)$

So, Odd perfect number doesn't exist.

P.S. Numbers which are of the form $(2^n-1)*2^{(n-1)}$ where (2^n-1) is a prime are perfect number.

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