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Perfect Cuboid

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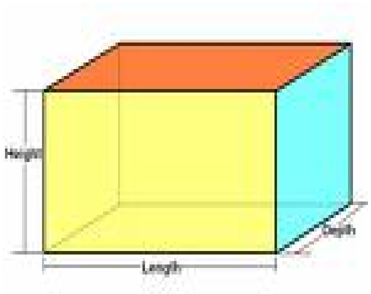
Abstract— **In mathematics, an Euler brick, named after Leonhard Euler, is a cuboid whose edges and face diagonals all have integer lengths. A primitive Euler brick is an Euler brick whose edge lengths are relatively prime.**

Index Terms— **Euler brick, cuboid, integer lengths.**

I. Introduction

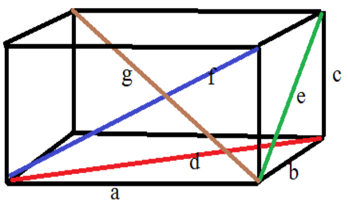
An Euler Brick is just a cuboid, or a rectangular box, in which all of the edges (length, depth, and height) have integer dimensions; and in which the diagonals on all three sides are also integers.



So if the length, depth and height are a, b, and c respectively, then a, b, and c are integers, as are the quantities $\sqrt{(a^2+b^2)}$ and $\sqrt{(b^2+c^2)}$ and $\sqrt{(c^2+a^2)}$.

The problem is to find a perfect cuboid, which is an Euler Brick in which the space diagonal, that is, the distance from any corner to its opposite corner, given by the formula $\sqrt{(a^2+b^2+c^2)}$*, is also an integer, or prove that such a cuboid cannot exist .*

II. Solution



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Let's the side of the cuboid be a,b,c.

The diagonals are d,e,f and the diagonal of the cuboid is g as shown in figure.

Now, $a^2 + b^2 = d^2$ (1)

$b^2 + c^2 = e^2$ (2)

$c^2 + a^2 = f^2$ (3)

$a^2 + b^2 + c^2 = g^2$ (4)

If, a is prime then $a^2 = d^2 - b^2$ from (1)

⇨ $a^2 = (d+b)(d-b)$

If a is prime then $d-b = 1$ and $d+b = a^2$

⇨ $d = (a^2+1)/2$ and $b = (a^2-1)/2$

Again from equation (2) $a^2 = (e+c)(e-c)$ = > $e-c=1$ and $e+c = a^2$

Again, $e = (a^2+1)/2$ and b $(a^2 -1)/2$

Implies b=c

So, $b^2 + c^2 = \sqrt{2}b$ which is contradiction.

So, a cannot be prime. Similarly, b and c cannot be prime.

So, we conclude that a, b, c none of them can be prime .... conclusion (1)

Now, a, b, c all are composite number. Let's say all are odd.

From (1), $a \equiv \pm 1 \pmod 4$

⇨ $a^2 \equiv 1 \pmod 4$

Similarly, $b^2 \equiv 1 \pmod 4$

And $d^2 \equiv 1 \pmod 4$

S0, left hand side $\equiv 1+1 = 2 \pmod 4$ whereas, right hand side $\equiv 1 \pmod 4$.

Here is contradiction.

So, a,b,c all cannot be odd ....conclusion (2)

Let's say a, b, c all are even.

Then d,e,f,g also even. So, there is a common factor 4 by which all equations will get divided. If still remains even then again all equations will be divided by 4 until one comes odd.

So, all cannot be even.........conclusion (3)

Let, two of a,b,c be odd and one even.

Let's say a,b odd and c even.

Now $a \equiv \pm 1 \pmod 4$

⇨ $a^2 \equiv 1 \pmod 4$

Similarly, $b^2 \equiv 1 \pmod 4$

From equation (1) left hand side is $\equiv 2 \bmod 4$. But right side must be even as odd + odd = even and right side is perfect square. So, it must be divisible by 4.

Here is the contradiction.

So, two of a,b,c cannot be odd and one cannot be even . ...conclusion (4)

From the above conclusions we can conclude that two of them must be even and one must be odd.

Let's say a,b are even and c is odd.

If we divide equation (1) by 4 gives remainder 0 on both sides.

- ⇨ The equation is divisible by 4. Now it will go on dividing by 4 at last it will give d as odd and one of a, b as odd and another even (divisible by 4).

Let's say $a = 4m^2$, $b = 16n^2$ and $d = 4p^2$ where m, n, p are odd.
Now, from equation (3), $c^2 + a^2 = f^2$
Now $c \equiv (\pm1 \text{ or } \pm3) \pmod 8$ (as c is odd)

- ⇨ $c^2 \equiv 1 \pmod 8$
- ⇨ $f^2 \equiv 1 \pmod 8$ (as f is also odd)
- ⇨ $m^2 \equiv 1 \pmod 8$ (as m is also odd)
- ⇨ $4m^2 \equiv 4 \pmod 8$
- ⇨ $a^2 \equiv 4 \pmod 8$

Now, if we divide equation (3) by 8 LHS gives remainder 4+1=5 and RHS 1
Contradiction.
So, m must be even. Implies $a^2 = 16m^2$ (putting 2m for m)
Now, if we divide both side of equation (2) by 16 LHS gives 0 whereas RHS gives a number (because $4p^2$ is not congruent to 0 mod 16 where p is odd)
Contradiction.
So, d^2 must be divisible by 16.
So, now $d^2 = 16p^2$ (putting 2p in place of p)
Accordingly $b^2 = 64n^2$ (putting 2n in place of n)
Now we have, $a^2 = 16m^2$, $b^2 = 64n^2$ and $d^2 = 16p^2$ (where m, n, p are odd)
Now, $c \equiv (\pm1, \pm3, \pm5, \pm7) \pmod{16}$

- ⇨ $c^2 \equiv 1 \text{ or } 9 \pmod{16}$
- ⇨ $f^2 \equiv 1 \text{ or } 9 \pmod{16}$

Now, if we divide both sides of equation (3) by 16 then we can conclude that c^2 and f^2 must give same remainder.
Let's say $c^2 = 16u + y$ (y =1 or 9)
$f^2 = 16v + y$.
Now, if we divide both side of equation (2) by 16 then we can conclude c^2 and e^2 should give same remainder as $b^2 \equiv 0 \pmod{16}$

- ⇨ $e^2 = 16w + y$.

Now, if we divide both sides of equation (2) by 32 then also c^2 and e^2 should give same remainder.
Now, $a^2 \equiv 16 \pmod{32}$ (as m is odd)
So, if we divide equation (3) by 32 then we get f^2 must be $\equiv (16+x) \pmod{32}$ where $c^2 \equiv x \pmod{32}$. Implies $e^2 \equiv x \pmod{32}$
Now we can write, $c^2 = 32u+y$; $e^2 = 32w+y$ and $f^2 = 32v+16+y$ (putting u=2u, w=2w, v=2v+1)
Now, if we divide equation (2) by 64 we get same remainder of c and e because $b^2 \equiv 0 \pmod{64}$
Therefore, we can write $c^2 = 64u+y$ and $e^2 = 64w+y$. (putting u=2u and w=2w)
Now, any odd integer can written as 4m±1
Now, $a^2 = 16(4m\pm1)^2$ (putting 4m±1 in place of m)

- ⇨ $a^2 = 16(16m^2\pm8m+1)$
- ⇨ $a^2 \equiv 16 \pmod{128}$

Now if we divide equation (3) by 128 LHS gives 16+64+y or (80+y) as remaider.
Now, RHS i.e. f^2 should give the same remainder on division by 128.
If we put v= 4v then $f^2 = 128v+16+y$

- ⇨ $f^2 \equiv 16+y \pmod{128}$ which doesn't match with LHS.

So, v must be odd.
Putting v = 4v±1 we get $f^2 = 32(4v\pm1)+16+y = 128v\pm32+16+y$ which doesn't give (80+y) as remainder.
Here is the contradiction.
Now, $a^2 = 64*(4\text{^}m)*m_1^2$, $b^2 = 64*4\text{^}(m+1)*m_2^2$ and $d^2 = 64*(4\text{^}m)*m_3^2$
Now, from equation (2), if we divide it by 64*4^(m+1) then c^2 and e^2 should give same remainder.
Say, $c^2 = 64*4\text{^}(m+1)*p_1+ p$ and $e^2 = 64*4\text{^}(m+1)*p_1 + p$
Now, from equation (4) $a^2 + b^2 = (g+c)(g-c)$
Now, LHS is divisible by 64*4^m. Therefore RHS also should get divided by it.
(g+c)(g-c) = 64*4^m*q*r
g+c must be equal to 2*64*4^(m-1)*q and g-c = 2r (any other combination will give c even which is contradiction).
Solving for c we get, c = 64*4^(m-1)*q - r

- ⇨ $c^2 = \{64*4\text{^}(m-1)*q - r\}^2$
- ⇨ $c^2 = 64^2*4\text{^}(2m-2)*q^2 - 2*64*4\text{^}(m-1)*r + r^2$

Equating both c^2 we get, $64*4\text{^}(m+1)*p_1 + p = 64^2*4\text{^}(2m-2)*q^2 - 2*64*4\text{^}(m-1)*r + r^2$
Or, $64*4\text{^}(m-1)\{16p_1 – 64*4\text{^}(m-1)*q^2 + 2r) + (p-r^2) = 0$
Now LHS has to be zero. As there are two terms and one is far bigger than the other (of remainders) so two terms independently should be zero.

- ⇨ $P = r^2$ and $16p_1 - 64*4\text{^}(m-1)*q^2 + 2r = 0$
- ⇨ $8p_1 - 32*4\text{^}(m-1)*q^2 +r =0$
- ⇨ $2[4p_1-16*4\text{^}(m-1)*q^2] + r = 0$

We se that the first term is even and r is odd. Now Difference of one even and one odd cannot give a zero. Here is the contradiction.

So, two of a,b,c cannot be even and one cannot be odd......conclusion (5)
From the above conclusion there is no such combination of a,b,c as far as a,b,c are integers.
So, Perfect cuboid doesn't exist where all sides and diagonals are integers.
Proved.

III. Conclusion

Perfect Cuboid doesn't exist.

References

[1] T.B. Bromhead, On square sums of squares, Math. Gaz., 44 (1960), p.219–220.
[2] E.Z. Chein, On the derived cuboid of an Eulerian triple, Canad. Math. Bull., Vol 20, no. 4 (1977), p. 509–510.
[3] W.J.A. Colman, On certain semiperfect cuboids, Fibonacci Quart., 26 (1988), p. 54-57. See also Some observations on the classical cuboid and its parametric solutions, p. 338-343.
[4] L.E. Dickson, History of the theory of numbers, Vol. 2, Diophantine Analysis, Carnegie Institute of Washington, (1919), Reprint by Chelsea, (1952); Ch. 15, ref. 28, p.448 and cross-refs to pp.446–458; Ch. 19, refs. 1–30, 40–45, pp. 497–502, 505–507.
[5] M. Gardner, Mathematical Games, Scientific American no. 1, 223 (July 1970), p.118; Correction, ibid. no. 3 (September 1970), p.218.
[6] R. Guy, Unsolved problems in number theory, Problem books in mathematics, Springer-Verlag, 1981.Korec, Nonexistence of

small Perfect Rational Cuboid, Acta Math. Univ. Comenian.42/43 (1983), p. 73–86.
[7] Korec, Nonexistence of small Perfect Rational Cuboid II, Acta Math. Univ. Comenian. 44/45 (1984), p. 39–48.
[8] Korec, Lower bounds for Perfect Rational Cuboids, Math. Slovaca, 42 (1992), No. 5, p.565-582.
[9] M. Kraitchik, On certain Rational Cuboids, Scripta Math. 11 (1945), p. 317–326.
[10] M. Kraitchik, Th´eorie des Nombres, Tome 3, Analyse Diophantine et application aux cuboides rationelles, Gauthier-Villars, Paris 1947, p. 122–131.
[11] M. Kraitchik, Sur les Cuboides rationelles, Proc. Int. Congr. Math., Amsterdam, 2 (1954), p. 33–34.
[12] J. Lagrange, Sur le cuboide entier, Seminaire DeLange-Pisot-Poitou (Groupe d'´etude de Th´eorie des Nombres), 17`eme ann´ee, 1975/1976, #G1.
[13] J. Lagrange, Sur le D´eriv´e du Cubo¨ıde Eul´erien, Canad. Math. Bull, Vol. 22 (2) (1979), p. 239–241.

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