On local existence and blow-up of solution for the higher-order nonlinear Kirchhoff-type equation with nonlinear strongly damped terms

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Abstract—In this paper, we deal with the initial boundary value problems for higher-order kirchhoff-type equation with nonlinear strongly dissipation:

At first, we prove the local existence and uniqueness of the solution by Galerkin method then and contracting mapping principle. Furthermore, we prove the global existence of solution, at last, we consider that blow up of solution in finite time under suitable condition.

Index Terms—Higher-order nonlinear Kirchhoff wave equation; local existence; The existence and uniqueness; blow-up

I. INTRODUCTION

In this paper we concerned with global existence and blow-up of solution for the following for Higher-order Kirchhoff-type equation with nonlinear strongly dissipation:

\[ u_t + (\Delta)^m u_t + \phi(\|\nabla u\|^2)(\Delta)^m u + h(u_t) = f(u) \quad \text{(1.1)} \]

\[ u(x,t) = 0, \quad i = 1,2, \ldots, m-1, x \in \partial \Omega, t > 0, \quad \text{(1.2)} \]

\[ u(x,0) = u_0, \quad u_i = u_i(x), \quad x \in \partial \Omega, \quad \text{(1.3)} \]

Where \( \Omega \subseteq \mathbb{R}^2 \) is bounded open domain with smooth boundary; \( \nabla \) is the outer norm vector; \( m > 1 \) is a positive integer, \( \phi(r) \) is a nonnegative locally Lipschitz, \( h(u_t) \) is a nonlinear forcing, \( f(u) \) is a nonlinear \( C^1 \)-function, \( (\Delta)^m u_t \) is a strongly dissipation. There have been many researches on the global and blow-up solution for Kirchhoff equation, we can see[1-8].

Kosuke Ono [9] deals with the higher-order kirchhoff-type equation with nonlinear dissipation:

\[ u_t + M(\|\nabla u\|^2)(\Delta u) + |u|^d u_t = f(u), \quad x \in \Omega, \quad t > 0, \quad \text{(1.4)} \]

\[ u(x,0) = u_0(x), \quad u_t(x,0) = u_i(x), \quad \text{and} \quad u(x,t)|_{\partial \Omega} = 0 \quad \text{(1.5)} \]

In where \( \Omega \) is a bounded domain in \( \mathbb{R}^n \) with smooth boundary \( \partial \Omega \), \( M(r) \) is a nonnegative \( C^1 \)-function for \( r \geq 0 \) positive satisfying \( M(\|\nabla u\|^2) = a + b\|\nabla u\|^{2\gamma} \) with \( a, b \geq 0, a + b > 0 \) and \( \gamma \geq 1 \), and \( f(u) \) constants is nonlinear \( C^1 \)-function satisfying \( |f(u)| \leq k_1|u|^{\alpha} \) and \( |f'(u)| \leq k_2|u|^\nu \) with \( k_1, k_2 > 0 \) and \( \alpha > 0 \).

In a bounded domain, where \( A = (\Delta)^m \), \( m > 1 \) is a positive integer, \( a, b, p > 0 \) and \( q, r > 2 \), are constants, we obtain the global existence of solutions by constructing a stable set in \( H_0^m(\Omega) \) and show the energy decay estimate by applying a lemma of V.Komornik.

Takeshi Taniguchi [11] considered the existence of local solution to a weakly damped wave equation of equation of kirchhoff type the damped term and the source term

\[ u_{tt}(t) - M(\|\nabla u(t)\|^2)\Delta u(t) - \gamma u_t(t) + |u(t)|^d u(t) = |u(t)|^q u(t) \quad \text{(1.9)} \]

\[ u(x,0) = u_0(x), \quad u_t(x,0) = u_i(x), \quad \text{and} \quad u(x,t)|_{\partial \Omega} = 0, \quad p, q, \gamma > 2 \quad \text{(1.10)} \]

with an initial value \( u(0,t)|_{\Omega} = 0 \quad \text{and} \quad u(0,t)|_{\Omega} = 0 \) and the Dirichlet boundary condition \( u_t(x,t)|_{\partial \Omega} = 0 \), where \( \Omega \) is an open bounded domain in \( \mathbb{R}^n \) with smooth boundary and \( M(r) \) is a locally Lipschitz function, he discuss the global existence and exponential asymptotic behavior of solution.

Recently, Gongwei Liu [12] concerned with the study of damped wave equation of kirchhoff type

\[ u_{tt} - M(\|\nabla u\|^2)\Delta u + u_t = g(u) \quad \text{(1.11)} \]
in $\Omega \times (0, \infty)$, with initial and Dirichlet boundary condition. Where $\Omega$ is the bound of $R^2$ have a smooth boundary $\partial \Omega$ under the assumption that $g$ is a function with exponential growth at infinity, he proves global existence and the decay property as well as blow-up of solutions in finite time under suitable conditions.

The paper is arranged as follows. In section 2, we state some preliminaries and some lemmas; in section 3, we obtain the existence and uniqueness of the local solution by the Banach contraction mapping principle; in section 4, we discuss global existence and nonexistence results; in section 5, we discuss the blow-up properties of solution for positive and negative initial energy and estimate for blow-up time $T^*$.

II. PRELIMINARIES

Throughout this paper, for convenience, we denote the norm and scalar product in $L^2(\Omega)$ by $\| \|$ and $(.,.)$; we use the next notations

$$(u,v) = \intuvdx_{\Omega}$$

$$(\|u\|^2 = \int|u|^2_{\Omega}$$

$$(\|u\| = \left(\int\int|u|^{2r}dx_{\Omega}\right)^{1/r}$$

where $C_i (i = 0 \cdots 12)$ are constants.

In this section, we present some materials needed in the proof of our results, we assume that there exists some constants

$$(G_1) \quad |f(u)| \leq k_1|u| + k_2|u|^{r+1}, \quad k_1, k_2, r > 0.$$

$$(G_2) \quad \text{there exist some constants}$$

(1) $h(s)s \geq 0, s \in \mathbb{R}$

(2) $|h(u)| \leq C_0(1 + \|\nabla^m u\|^{1-a}), \quad 0 < a < 1.$

(3) $|h(u)| \leq k_3|u|^{p+1}, \quad k_3, p \geq 0.$

$$(G_3) \quad \text{there exist a constant} \quad \beta > 0, \quad \text{such that}$$

$$(u,v) \quad f(u) \geq (2 + 4\beta)G(s) \quad \text{for all} \quad s \in \mathbb{R}.$$ 

In this paper, we will use the next well-know Lemmas Lemma 2.1[10] Let $s$ be a number with $2 \leq s < +\infty, n \leq 2m$, then there is a constant $C$ depending on $\Omega$ and $s$ such that

$$\|u\| \leq \kappa_{s}(-\Delta)^{\frac{m}{2s}} \quad \forall u \in H^{m}_0(\Omega) \quad \text{(2.1)}$$

Lemma 2.2[13](young inequality) Let $a, b$ and $\mu$ are positive constants, such that $0 \leq r < 2m \quad \text{and} \quad n - 2m$ we have

$$(ab \leq \frac{\mu r^p}{p} + \frac{1}{q\mu^r}b^q \quad \left(\frac{1}{p} + \frac{1}{q} = 1, \quad p > 1, q > 1\right) \quad \text{(2.2)}$$

Lemma 2.3[14] Suppose that $\delta > 0$ and $B(t)$ is a nonnegative $C^{2}(0, +\infty)$ function such that

$$B'(t) - 4(\delta + 1)B(t) + 4(\delta + 1)B(t) \geq 0. \quad \text{(2.3)}$$

If

$$B(0) > r_2B(0) + K_0 \quad \text{(2.4)}$$

Then we have $\forall t > 0, B(t) > K_0$ where $K_0$ is a constant $r_2 = 2(\delta + 1) - 2\sqrt{(\delta + 1)\delta}$ is smaller root of the equation

$$r^2 - 4(\delta + 1)r + 4(\delta + 1) = 0 \quad \text{(2.5)}$$

Lemma 2.4[14] If $J(t) = \text{non-increasing function}$ on $[t_0, +\infty), t_0 \geq 0$ such that

$$J'(t)^2 \geq a + bJ(t)^{2+1/\beta}, \quad \forall t_0 \geq 0, \quad \text{(2.6)}$$

where $a > 0, b \in \mathbb{R}$.

Then there exists a finite time $T^*$ such that

$$\lim_{t \to T^*} J(t) = 0 \quad \text{(2.7)}$$

And the case that

(i) If $b > 0$,

$$J(t_0) < \min \{1, \sqrt{\frac{a}{b}}\} \quad \text{then}$$

$$T^* \leq t_0 + \frac{1}{\sqrt{-b}} \ln \frac{\sqrt{a}}{-b} - J(t_0) \quad \text{(2.8)}$$

(ii) If $b = 0$,

$$T^* \leq t_0 + \frac{\sqrt{a}}{\sqrt{-b}} \quad \text{or}$$

$$T^* \leq \frac{J(t_0)}{\sqrt{-b}} \quad \text{(2.9)}$$

(iii) If $b > 0$,

$$T^* \leq \frac{J(t_0)}{\sqrt{a}} \quad \text{(2.10)}$$

III. EXISTENCE OF LOCAL SOLUTION

In this section, we prove the existence of local solution to problem (1.1)-(1.3) for initial value $(u_0, u_1) \in H^m(\Omega) \cap H^{2m}_{H^m}(\Omega) \times H^m(\Omega)$.

Theorem 3.1 Suppose that $(G_1), (G_2)$ hold, for any give $(u_0, u_1) \in H^{2m}(\Omega) \cap H^m(\Omega) \times H^m(\Omega)$.
\[ u(t) \in C([0,T_0]; H^{m}_0(\Omega) \cap H^2(\Omega)) \]
\[ u_t(t) \in C([0,T_0]; L^2(\Omega) \cap H^m(\Omega)) \]

Proof we set for any \( T > 0 \), the Banach space
\[ X_{T,R} = \left\{ v(t) \in C([0,T]; H^m_0(\Omega) \cap H^2(\Omega)), \right\} \]
\[ \rho(v(t)) \leq R. \]

Where \( \rho(v(t)) = \|v_t\|^2 + \|(-\Delta)^m v\|^2 \). Define distance \( \rho(u,v) = \sup_{t \leq T} \rho(u(t) - v(t)) \). Then \( X_{T,R} \)

Is a complete metric space.

We define map the \( \mathcal{X}: v \mapsto \mathcal{X}(v) = u \). \( u \) is the unique solution of the following equation:
\[ u_{tt} + (-\Delta)^m u + \phi(\|\nabla u\|^2)(-\Delta)^m u + h(u_t, u_t) = f(v) \]
(3.1)
\[ u(x,t) = 0, \quad \frac{\partial u(x,t)}{\partial \nu} = 0, \quad i = 1,2, \cdots, m-1, \quad x \in \partial \Omega, t \geq 0. \]
(3.2)
\[ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega. \]
(3.3)

we define map the \( T > 0 \) and \( R > 0 \), such that

(1) \( \mathcal{X} \) maps \( X_{T,R} \) into itself;
(2) \( \mathcal{X} \) is a contraction mapping with respect to the metric \( d(\cdot, \cdot) \).

Step 1. we will show that \( \mathcal{X} \) maps \( X_{T,R} \) into itself, we multiply \( u_t + \varepsilon(-\Delta)^m u \) with both sides of equation (3.4) and obtain
\[ (u_{tt} + (-\Delta)^m u_t + \phi(\|\nabla u\|^2)(-\Delta)^m u_t + h(u_t, u_t) + (-\Delta)^m u_t) = (f(v), u_t + (-\Delta)^m u) \]
(3.4)
\[ (u_{tt}, u_t + \varepsilon(-\Delta)^m u) \]
\[ = \frac{1}{2} \frac{d}{dt} \|u_t\|^2 + \varepsilon \frac{d}{dt} \|(-\Delta)^m u\|^2 \]
(3.5)
\[ ((-\Delta)^m u_t, u_t + \varepsilon(-\Delta)^m u) \]
\[ = \|\nabla^m u_t\|^2 + \varepsilon \frac{d}{dt} \|(-\Delta)^m u\|^2 \]
(3.6)
\[ (\phi(\|\nabla v\|^2)(-\Delta)^m u_t, u_t + \varepsilon(-\Delta)^m u) \]
\[ = \frac{1}{2} \frac{d}{dt} (\phi(\|\nabla v\|^2) \|\nabla v\|^2) \]
\[ - \frac{1}{2} \frac{d}{dt} (\phi(\|\nabla v\|^2) \|\nabla v\|^2 + \varepsilon \phi(\|\nabla v\|^2) \|(-\Delta)^m u\|^2) \]
(3.7)
\[ \left| \frac{d}{dt} (\phi(\|\nabla v\|^2) \|\nabla v\|^2) \right| \]
\[ \leq \|\phi(\varepsilon)\|_{\infty} \|(-\Delta)^m v\| \|\nabla v\|^2 \]
\[ \leq LR \|\nabla u\|^2 \]
\[ \leq \frac{LR}{\lambda^m} \|(-\Delta)^m u\|^2 \]
\[ \leq \kappa_0 \rho(u(t)) \]
(3.8)
On local existence and blow-up of solution for the higher-order nonlinear Kirchhoff-type equation with nonlinear strongly damped terms

\[
(\phi(\|\nabla^m u\|)(\Delta)^m u, u_t + \epsilon(\Delta)^m u) \\
\geq \frac{1}{2} \frac{d}{dt} (\phi(\|\nabla^m u\|)\|\nabla^m u\|^2) \\
- \kappa_0 \rho(u(t)) + \epsilon \phi(\|\nabla^m u\|^2) (-\Delta)^m u \\
= (h(u_t), u_t) + \epsilon (h(u_t), (-\Delta)^m u) 
\] \tag{3.9} 

According to young inequality and \(G_2\), such that

\[
(h(u_t), \epsilon (-\Delta)^m u) \geq -\frac{\epsilon}{2} \|h(u_t)\|^2 - \frac{\epsilon}{2} \|(-\Delta)^m u\|^2 
\] \tag{3.11} 

\[
\|h(u_t)\|^2 \\
\leq C_0 (1 + \|\nabla^m u\|^2) \\
\leq 2C_0 + 2\epsilon \|\nabla^m u\|^2 \\
(h(u_t), (-\Delta)^m u) \\
\geq -C_0 - \epsilon \|\nabla^m u\|^2 - \frac{\epsilon}{2} \|(-\Delta)^m u\|^2 
\] \tag{3.12} 

According to Lemma 2.1 and \((G_1)\), such that

\[
(f(v), u, \epsilon (-\Delta)^m) \\
\leq \int (k_1 |v| + k_2 |v|^r) u dx + \epsilon \int (k_1 |v| + k_2 |v|^r) (-\Delta)^m u dx \\
\leq (k_1 \|v\| + k_2 \|v\|^r) \|u\| + \epsilon (k_1 |v| + k_2 |v|^r) \|(-\Delta)^m u\| \\
\leq (\frac{k_1 \sqrt{R}}{\lambda^{2m}} + k_2 \kappa R^{r+1}) \|u\| + (\frac{\epsilon k_1 \sqrt{R}}{\lambda^{2m}} + \epsilon k_2 \kappa R^{r+1}) \|(-\Delta)^m u\| 
\] \tag{3.14} 

\[
\alpha_0 = \{\frac{k_1 \sqrt{R}}{\lambda^{2m}} + k_2 \kappa R^{r+1}, \frac{\epsilon k_1 \sqrt{R}}{\lambda^{2m}} + \epsilon k_2 \kappa R^{r+1}\} \\
\leq \alpha_0 (\|u\| + \|(-\Delta)^m u\|) \\
\leq \alpha_0 (\rho(u))^\frac{1}{2} 
\] \tag{3.15} 

From (3.5) and (3.15), we have

\[
\frac{1}{2} \frac{d}{dt} (\|u\|^2 + 2\epsilon (u, (-\Delta)^m u) + \epsilon \|(-\Delta)^m u\|^2 + \phi(\|\nabla^m u\|^2 \|\nabla^m u\|^2)) \\
+ \|\nabla^m u\|^2 - \epsilon \|\nabla^m u\|^2 \geq - C_0 \epsilon \|\nabla^m u\|^2 - \frac{\epsilon}{2} \|(-\Delta)^m u\|^2 \\
\leq \alpha_0 (\rho(u))^\frac{1}{2} + \kappa_0 \rho(u(t)) + C_0 
\] \tag{3.16} 

\[
\frac{dt}{d} (\frac{1}{2} \|u\|^2 + \epsilon (u, (-\Delta)^m u) + \|(-\Delta)^m u\|^2 + \frac{1}{2} \phi(\|\nabla^m u\|^2 \|\nabla^m u\|^2) + K_1) \\
+ (1 - \epsilon - \frac{C_0 \epsilon}{2}) \|\nabla^m u\|^2 + \epsilon \phi(\|\nabla^m u\|^2 \|(-\Delta)^m u\|^2) 
\]
\[
\rho_1(u(t)) = \|u_t\|_2^2 + 2\varepsilon(u, (-\Delta)^m u) + \varepsilon(-\Delta)^m u_0^2 + \phi(\|\nabla^m v\|_2^2)\|\nabla^m u\|_2^2 + 2K_1
\]

such that
\[
\|u_t\|_2^2 + 2\varepsilon(u, (-\Delta)^m u) + \varepsilon(-\Delta)^m u_0^2 + \phi(\|\nabla^m v\|_2^2)\|\nabla^m u\|_2^2 + 2K_1
\geq (1 - 2\varepsilon)\|u_t\|_2^2 + \frac{\varepsilon}{2}\|(-\Delta)^m u\|_2^2
\]

From (3.18) and (3.19), we have
\[
\|u_t\|_2^2 + 2\varepsilon(u, (-\Delta)^m u) + \varepsilon(-\Delta)^m u_0^2 + \phi(\|\nabla^m v\|_2^2)\|\nabla^m u\|_2^2 + 2K_1
\]

We take \(\alpha_1 = \min\{1 - \frac{\varepsilon}{2}\}\), such that
\[
\rho_1(u(t)) \geq \alpha_1(\|u_t\|_2^2 + \|(-\Delta)^m u\|_2^2)
\]

So
\[
\rho_1(u(t)) \geq \alpha_1 \rho(u(t))
\]

\[
\frac{dt}{d}\left(\|u_t\|_2^2 + 2\varepsilon(u, (-\Delta)^m u) + \varepsilon(-\Delta)^m u_0^2 + \phi(\|\nabla^m v\|_2^2)\|\nabla^m u\|_2^2 + 2K_1\right) + \alpha_1\|\nabla^m u\|_2^2
\]

\[
\leq \alpha_0(\rho(u))^\frac{1}{2} + \kappa_0 \rho(u(t)) + \frac{\varepsilon}{2} \rho(u(t)) + C_0
\]

Where
\[
\alpha_2 = 1 - \varepsilon - \frac{\rho u_0}{2}, \quad C_0 \leq \frac{2\kappa_0 K_1}{\alpha_1}
\]

We take
\[
\frac{d}{dt}\rho_1(u(t)) + \alpha_2\|\nabla^m u\|_2^2
\]

\[
\leq \frac{2\kappa_0}{\alpha_1} \rho_1(u(t)) + \frac{\varepsilon}{2} \rho_1(u(t)) + \alpha_0(\rho_1(u))^\frac{1}{2}.
\]

So, using Gronwall inequality, we obtain
\[
\rho_1(u(t)) + \alpha_2 \int_0^T \|\nabla^m u\|_2^2 ds
\]

\[
\leq \left\{\rho_1(u(0))^\frac{1}{2} + \alpha_0 T\right\}^2 \rho\left(\frac{(4\kappa_0 + \varepsilon) T}{2\alpha_1}\right).
\]

Where
\[
\rho_1(u(0)) = \|u_t\|_2^2 + 2\varepsilon(u, (-\Delta)^m u_0) + \varepsilon(-\Delta)^m u_0^2 + \phi(\|\nabla^m v\|_2^2)\|\nabla^m u_0\|_2^2 + 2K_1
\]

From we obtain
\[
\rho(u(t)) + \alpha_2 \int_0^T \|\nabla^m u\|_2^2 ds
\]

\[
\leq \frac{1}{\alpha_1} \rho_1(u(t)) + \frac{\alpha_2}{\alpha_1} \int_0^T \|\nabla^m u\|_2^2 ds
\]
On local existence and blow-up of solution for the higher-order nonlinear Kirchhoff-type equation with nonlinear strongly damped terms

\[
\frac{1}{\rho_1(u(0))^2 + \alpha_0 T^2} \rho \frac{(4k_{n}+\lambda_{0}\alpha_{T})T}{2\alpha_0}.
\]

Therefore, we choose that parameters \(T\) and \(R\), we obtain

\[
\{\rho_1(u(0))^2 + \alpha_0 T^2 \rho \frac{(4k_{n}+\lambda_{0}\alpha_{T})T}{2\alpha_0} \leq R^2.
\]

So, we finish the proof of the first step.

Step 2 we will show that \(\mathcal{X}\) is a contraction in \(X_{T,R}\), let \(v_1, v_2 \in X_{T,R}\), such that \(\mathcal{X}(v_i) = u_i\)

\[
\mathcal{X}(v_2) = u_2. \text{ Setting } w = u_1 - u_2, \text{ then we have}
\]

\[
w_i + (-\Delta)^m w_i + \phi(\|\nabla^m v_i\|^2)((-\Delta)^m) u_i - \phi(\|\nabla^m v_2\|^2)((-\Delta)^m) u_2) + h(u_i) - h(u_2) = f(v_i) - f(v_2)
\]

\[
\begin{align*}
&\frac{1}{2} d \| w_i \|^2 + \alpha_0 \| \nabla^m w_i \|^2 - \varepsilon \| \nabla^m w_i \|^2 \\
&\frac{1}{2} d \| ((-\Delta)^m) u_i, w_i, \| \varepsilon ((-\Delta)^m) w_i \|
\end{align*}
\]

\[
(\phi(\|\nabla^m v_1\|^2))((-\Delta)^m) u_1 - \phi(\|\nabla^m v_2\|^2))((-\Delta)^m) u_2) + \varepsilon ((-\Delta)^m) w_i
\]

\[
= \frac{1}{2} d \phi(\|\nabla^m v_1\|^2)\|\nabla^m w_i\|^2 + \phi(\|\nabla^m v_1\|^2)\|\nabla^m w_i\|^2 - \frac{1}{2} d \phi(\|\nabla^m v_1\|^2)\|\nabla^m w_i\|^2
\]

\[
+ \frac{1}{2} d \phi(\|\nabla^m v_1\|^2)\|\nabla^m w_i\|^2 + \varepsilon \frac{d}{2} \|\nabla^m w_i\|^2
\]

\[
\geq \frac{1}{2} d \phi(\|\nabla^m v_1\|^2)\|\nabla^m w_i\|^2 + \varepsilon \frac{d}{2} \|\nabla^m w_i\|^2
\]

\[
\geq \frac{1}{2} d \phi(\|\nabla^m v_1\|^2)\|\nabla^m w_i\|^2 + \varepsilon \frac{d}{2} \|\nabla^m w_i\|^2
\]

\[
\geq \frac{1}{2} d \phi(\|\nabla^m v_1\|^2)\|\nabla^m w_i\|^2 + \varepsilon \frac{d}{2} \|\nabla^m w_i\|^2
\]
\[-L_3[\rho(v_1 - v_2)]^{\frac{1}{2}}[\rho(w)]^{\frac{1}{2}} - L_0\rho(w(t))\]  

(3.31)

From \(G_2\) we have

\[(h(u_1) - h(u_2), w_1 + \varepsilon(-\Delta)^m w)\]

\[\leq L_4\|w_1\|^2 + L_5\|(-\Delta)^m w\|^2\]

\[\leq \beta_0 \rho(w(t))\]  

(3.32)

Where \(\beta_0 = \max\{L_4, L_5\}\).

From \(G_1\) and Lemma 2.1 we have

\[(f(v_1) - f(v_2), w_1 + \varepsilon(-\Delta)^m w)\]

\[\leq \int_\Omega k_1|v_1 - v_2|^r|w_1| + k_2|v_1|^r - |v_2|^r|w_1|\,dx\]

\[+ \int_\Omega k_1|v_1 - v_2|^r|w_1| + k_2|v_1|^r - |v_2|^r|(-\Delta)^m w|\,dx\]

\[\leq (k_1 + k_2\|v_1\|_r + |v_2|^r + |v_2|^r)(|v_1 - v_2|_2 + \frac{\varepsilon}{n^2}(\Delta)^m w)\]

\[+ L_6\|(-\Delta)^m (v_1 - v_2)\|_2\|w_1\| + L_7\|(-\Delta)^m (v_1 - v_2)\|_2\|(-\Delta)^m w\|\]

\[\leq L_6\rho(v_1 - v_2)^{\frac{1}{2}}[\rho(w)]^{\frac{1}{2}} + L_7\rho(v_1 - v_2)^{\frac{1}{2}}[\rho(w)w(t)]^{\frac{1}{2}}\]  

(3.33)

From (3.29)-(3.33), we obtain

\[\frac{d}{dt}\left(\frac{1}{2}\|w_1\|^2 + \varepsilon\frac{1}{2}\|(-\Delta)^m w\|^2 + \varepsilon\frac{1}{2}\|w_1\|^2 + \varepsilon\|(-\Delta)^m w\|^2\right)\]

\[\leq \frac{1}{2}\|\nabla w_1\|^2 + \varepsilon\|\nabla w_1\|^2 - L_5\rho(v_1 - v_2)^{\frac{1}{2}}[\rho(w)]^{\frac{1}{2}} - L_0\rho(w(t)) - \beta_0\rho(w(t))\]

\[\leq L_6\rho(v_1 - v_2)^{\frac{1}{2}}[\rho(w)]^{\frac{1}{2}} + L_7\rho(v_1 - v_2)^{\frac{1}{2}}[\rho(w)w(t)]^{\frac{1}{2}}\]  

(3.34)

From (3.34) we obtain

\[\frac{d}{dt}\left(\|w_1\|^2 + \varepsilon\|\nabla w_1\|^2 + \varepsilon\|(-\Delta)^m w\|^2 + 2\varepsilon\|w_1\|^2 + \varepsilon\|(-\Delta)^m w\|^2\right)\]

\[\leq (L_0 + \beta_0)\rho(w(t)) + (L_6 + L_4 + L_5)[\rho(v_1 - v_2)]^{\frac{1}{2}}[\rho(w)]^{\frac{1}{2}}\]  

(3.35)

Where we set

\[\rho_2(w(t)) = \|w_1\|^2 + \varepsilon\|\nabla w_1\|^2 + \varepsilon\|(-\Delta)^m w\|^2 + 2\varepsilon\|w_1\|^2 + \varepsilon\|(-\Delta)^m w\|^2\]  

(3.36)

\[2\varepsilon\|w_1\|^2 + \varepsilon\|(-\Delta)^m w\|^2 + 2\varepsilon\varepsilon(w_1, (-\Delta)^m w)\]

\[\leq \frac{\varepsilon}{2}\|(-\Delta)^m w\|^2\]  

(3.37)

Then we have

\[\rho_2(w(t))\]

\[\leq \|w_1\|^2 + \varepsilon\|(-\Delta)^m w\|^2 + 2\varepsilon(w_1, (-\Delta)^m w)\]

\[\geq \|w_1\|^2 + \varepsilon\|(-\Delta)^m w\|^2 + 2\varepsilon(w_1, (-\Delta)^m w)\]
On local existence and blow-up of solution for the higher-order nonlinear Kirchhoff-type equation with nonlinear strongly damped terms

\[ \geq (1 - 2\varepsilon)\|w\|^2 + \frac{\varepsilon}{2} \|(-\Delta)^m w\|^2 \]

\[ \geq \beta_1 (\|w\|^2 + \|(-\Delta)^m w\|^2) \]

Where

\[ \beta_1 = \min\{1 - 2\varepsilon, \frac{\varepsilon}{2}\} \]

\[ \frac{d\rho_2(w(t))}{dt} \]

\[ \leq (L_0 + \beta_0)\rho(w(t)) + (L_3 + L_8 + L_9)\left(\frac{1}{\rho(w)}\right)^\frac{1}{2}\|(-\nabla)^m w\|^2 \]

(3.39)

\[ \leq (L_0 + \beta_0)\rho_2(w(t)) + (L_3 + L_8 + L_9)\left(\frac{1}{\rho_2(w(t))}\right)^\frac{1}{2}\|(-\nabla)^m w\|^2 \]

(3.40)

Where \( \rho_2(w(0)) = 0 \).

Therefore, we use Gronwall inequality, we have

\[ \rho_2(w(t)) \leq \left(\frac{L_0 + L_8 + L_9}{L_2} + L_3\right)^2 T^2 e \sup_{0 \leq \tau \leq T} \rho(v_1(t) - v_2(t)) \]

(3.41)

So, we have

\[ \sup_{0 \leq \tau \leq T} \rho(u_1(t) - u_2(t)) \leq C_{T,R} \sup_{0 \leq \tau \leq T} \rho(v_1(t) - v_2(t)) \]

(3.42)

Where \( C_{T,R} = \left(\frac{L_0 + L_8 + L_9}{L_2} + L_3\right)^2 T^2 e \frac{(L_0 + \beta_0)T}{\beta_1} \).

It is easy to see that \( C_{T,R} < 1 \) for a small \( T > 0 \).

we finish the proof of the second step. By applying the Banach contraction mapping theorem, the proof of Theorem 3.1 is now complete.

IV. THE BLOW-UP IN FINITE TIME

In this section, we consider the blow-up of solutions, we assume that \( \Phi(s) = s^q \), \( q \geq 0 \), \( h(u) = u \), such problems (1.1)-(1.3) became

\[ u_t(t) + (-\Delta)^m u_t(t) + \|\nabla^m u\|^2 (-\Delta)^m u(t) + u_t(t) = f(u(t)) \]

(4.1)

Next, we define the energy function of the solution \( u \) of (4.1)

\[ E(t) = \frac{1}{2}\|u_t\|^2 + \frac{1}{2(q + 1)}\|\nabla^m u\|^{2(q + 1)} - \int_{\Omega} G(u) dx, \]

(4.2)

\[ G(s) = \int_{0}^{s} g(s) ds. \]

where

Then, we have

\[ E(t) = E(0) - \int_{0}^{t} (\|\nabla^m u\|^2 + \|u_t\|^2) ds, \]

(4.3)

\[ E(0) = \frac{1}{2}\|u_t\|^2 + \frac{1}{2(q + 1)}\|\nabla^m u_t\|^{2(q + 1)} - \int_{\Omega} G(u_0) dx. \]

where

Definition 4.1 A solution \( u(t) \) of (4.1) is called a blow-up solution if there exist a finite time \( T^* \) such that

\[ \lim_{t \to T^*} \int_{\Omega} (\|\nabla^m u\|^2 + \|u_t\|^2) dx = +\infty. \]

(4.4)

For the next lemma, we define
\[ Y(t) = Y(u(t)) = \|u(t)\|^2 + \frac{1}{2} \left( \|u(s)\|^2 + \|\nabla^n u(s)\|^2 \right) ds \quad \text{for} \quad t \geq 0. \quad (4.5) \]

Lemma 4.1 Assume \((G_1), (G_3)\) hold, Let

\[ Y'(t) - 4(\beta + 1)\|u_t\|^2 \geq (-4 - 8\beta)E(0) + (4 + 8\beta) \left( \|u_t\|^2 + \|\nabla^n u_t(s)\|^2 \right) ds. \quad (4.6) \]

Proof. From (4.5), we have

\[ Y'(t) = 2 \int \Omega uu_t dx + \|u(t)\|^2 + \|\nabla^n u\|^2, \quad (4.7) \]

and

\[ Y'(t) = 2\|u(t)\|^2 - 2\|\nabla^n u\|^{2q+2} + 2\int \Omega f(u)udx. \quad (4.8) \]

From above equation and the energy identity, we have

\[ Y'(t) - 4(\beta + 1)\|u_t\|^2 = (-4 - 8\beta)E(0) + (4 + 8\beta) \int_0^t \left( \|u_t\|^2 + \|\nabla^n u_t\|^2 \right) ds \]

\[ + \int_\Omega (2f(u) - (4 + 8\beta)G(u))dx + \frac{2 + 4\beta}{q + 1} \|\nabla^n u\|^{2q+1} - 2\|\nabla^n u\|^{2q+2}, \quad (4.9) \]

there form the assumption \((G_3)\), we have

\[ 2\int_\Omega (f(u) - (2 + 4\beta)G(u))dx + \left( \frac{2 + 4\beta}{q + 1} \|\nabla^n u\|^{2q+1} - 2\|\nabla^n u\|^{2q+2} \right) \geq 0. \quad (4.10) \]

So, we obtain (4.6)

Now, we can consider there different cases on the sign of initial energy \(E(0)\)

If \(E(0) < 0\), from (4.6), we have

\[ Y'(t) \geq (-4 - 8\beta)E(0), \quad (4.11) \]

integration (4.11) over \([0, t]\), we have that

\[ Y'(t) \geq Y'(0) - 4(1 + 2\beta)E(0)t. \quad (4.12) \]

Thus, we get

\[ Y'(t) > \|u_0\|^2 + \|\nabla^n u_0\|^2 \quad \text{for} \quad t > t^*, \quad \text{where} \]

\[ t^* = \max\left\{ \frac{Y'(0) - (\|u_0\|^2 + \|\nabla^n u_0\|^2)}{4(1 + 2\beta)E(0)} \right\}. \quad (4.13) \]

If \(E(0) = 0\), from (4.6) we have

\[ Y' \geq 4(\beta + 1)\|u_t\|^2 + (4 + \beta) \int_0^t \left( \|u_t\|^2 + \|\nabla^n u_t\|^2 \right) ds \geq 0 \quad t \geq 0 \quad (4.14) \]

Integration (4.9) over \([0, 1]\), we have

\[ Y \geq Y(0) \quad (4.15) \]

Furthermore, if

\[ Y'(0) > \|u_0\|^2 + \|\nabla^n u_0\|^2 \quad i.e. \int\Omega uu_t dx > 0 \quad \text{so} \]

\[ Y'(t) \geq Y'(0) > \|\nabla^n u_0\|^2 + \|u_0\|^2 \quad \text{for} \quad t \geq 0. \quad (4.16) \]

If \(E(0) > 0\) we first note that
On local existence and blow-up of solution for the higher-order nonlinear Kirchhoff-type equation with nonlinear strongly damped terms

\[ 2\left( \int_0^t uu_r dxdt + \int \nabla u \nabla u_r ds \right) = \|u_t\|^2 + \|\nabla u_t\|^2 - (\|u_0\|^2 + \|\nabla u_0\|^2). \]  

(4.10)

By using Holder inequality and (4.10) we have

\[ \|u(t)\|^2 + \|\nabla u(t)\|^2 \leq \|u_0\|^2 + \|\nabla u_0\|^2 + \int_0^t \|u_r(s)\|^2 ds + \int_0^t \|\nabla u_r(s)\|^2 ds + \int_0^t \|u_t(s)\|^2 ds \]

so, from above, we obtain

\[ Y(t) = 2\int_0^t uu_r dx + \|u(t)\|^2 + \|\nabla u(t)\|^2 \leq \|u_0\|^2 + \|\nabla u_0\|^2 + \int_0^t \|u_r(s)\|^2 ds + \int_0^t \|\nabla u_r(s)\|^2 ds + \int_0^t \|u_t(s)\|^2 ds \]

(4.11)

\[ Y'(t) \leq Y(t) + \|u_r(t)\|^2 + \|\nabla u_t(t)\|^2 + \|u_t(t)\|^2 \]

(4.12)

Form above inequality and (4.6), we obtain

\[ Y(t) - 4(\beta + 1)Y(t) + 4(1 + \beta)Y(t) + Y_1 \geq 0. \]  

(4.14)

Where

\[ Y_1 = (4 + 8\beta)E(0) + 4(1 + \beta)(\|u_0\|^2 + \|\nabla u_0\|^2). \]  

(4.18)

Setting

\[ B(t) = Y(t) + \frac{Y_1}{4(1 + \beta)} \quad t > 0 \]

(4.19)

Then \( B(t) \) satisfies Lemma 2.3, if

\[ Y'(0) > r_2[Y(0) + \frac{Y_1}{4(1 + \beta)}] + \|u_0\|^2 + \|\nabla u_0\|^2 \]

(4.20)

Then

\[ Y'(0) > \|u_0\|^2 + \|\nabla u_0\|^2 \quad \text{for all} \quad t > 0 \]

Lemma 4.2 Assume that \( (G_1) \) and \( (G_2) \) hold and that either one of the following conditions is satisfied

\[ E(0) < 0; \]

\[ E(0) = 0 \quad \text{and} \quad \int_\Omega u_0u_1 dx > 0; \]

\[ E(0) > 0 \quad \text{and} \quad (4.20) \quad \text{holds, then} \quad Y'(0) > \|u_0\|^2 + \|\nabla u_0\|^2 \quad \text{for all} \quad t > 0 \]

where \( t = t^* \) is given by (4.13) in case (1): \( t_0 = 0 \) in cases (2) and (3).

Proof. We can prove this Lemma 4.2 by Lemma 4.1.

Theorem 4.1 Under the assumptions \( (G_1) \) and \( (G_2) \), and that either one of the following condition is satisfied:

\[ E(0) < 0; \]

\[ E(0) = 0 \quad \text{and} \quad \int_\Omega u_0u_1 dx > 0; \]

\[ 0 < E(0) < \frac{(\int_\Omega u_0u_1 dx)^2}{2(T_1 + 1)(\|\nabla u_0\|^2 + \|u_0\|^2)} , \quad \text{and by (4.20) hold} . \]

where \( T_1 \) to be chosen later.
Then the solution \( u \) blow-up at finite \( T^* \), and \( T^* \) can be estimate by (4.35)-(4.38), respectively, according to the sign of \( E(0) \).

Proof. Let
\[
J(t) = (Y(t) - \left(\|u_0\|^2 + \|\nabla u_0\|^2\right) - \frac{1}{\beta}) \quad t \in [0, T]
\]
(4.21)

where \( T^* \) is some certain constant which will be chosen later. Then we get
\[
J(t) = -\beta J(t) \left(\frac{1}{\beta} \left( Y(t) + (T_1 - t)\left(\|u_0\|^2 + \|\nabla u_0\|^2\right)\right)\right)
\]
(4.22)

And
\[
J'(t) = -\beta J(t) \left(\frac{2}{\beta} V(t)\right)
\]
(4.23)

\[
V(t) = Y(t)\left[ Y(t) + (T_1 - t)\left(\|u_0\|^2 + \|\nabla u_0\|^2\right)\right] - (1 + \beta)\left( Y(t) - \left(\|u_0\|^2 + \|\nabla u_0\|^2\right)\right)
\]
(4.24)

Next we denote
\[
P = \|u(t)\|^2, \quad Q = \int_0^t \|\nabla u(s)\|^2 ds, \quad R = \|u_0(t)\|^2, \quad S = \int_0^t \|\nabla u_0(s)\|^2 ds
\]
(4.25)

By (4.7) and (4.9), and Holder inequality we have
\[
Y(t) = 2\int_{\Omega} u(t)u_i(t)dx + \|u_0\|^2 + \|\nabla u_0\|^2 + 2\int_0^t \int_{\Omega} (u(s)u_i(s) + \nabla^m u(s)\nabla^m u_i(s))dxds
\]
\[
\leq \|u_0\|^2 + \|\nabla u_0\|^2 + 2\|u(t)\|\|u_i(t)\| + 2(\lambda^{-2m} + 1) \left(\int_0^t \|\nabla u_0\|^2 ds\right)^{\frac{1}{2}} \left(\int_0^t \|\nabla u_0\|^2 ds\right)^{\frac{1}{2}}
\]
\[
\leq \|u_0\|^2 + \|\nabla u_0\|^2 + 2\sqrt{PR} + 2(\lambda^{-2m} + 1)\sqrt{QS}
\]
\[
\leq \|u_0\|^2 + \|\nabla u_0\|^2 + \gamma (\sqrt{PR} + \sqrt{QS})
\]
(4.26)

where \( \gamma = 2 + 2\lambda^{-2m} \)

From (4.6), we have
\[
Y(t) \geq (-4 - 8\beta)E(0) + (4 + 8\beta)(R + S)
\]
(4.27)

From (4.24)-(4.27), we have
\[
V(t) \geq \left[ (-4 - 8\beta)E(0) + 4(1 + \beta)(R + S)\right] \left[ (Y(t) + (T_1 - t)\left(\|u_0\|^2 + \|\nabla u_0\|^2\right)\right) - (1 + \beta)\left( Y(t) - \left(\|u_0\|^2 + \|\nabla u_0\|^2\right)\right)
\]
\[
\geq (-4 - 8\beta)E(0)J(t) \left(\frac{1}{\beta}\right) + 4(1 + \beta)(R + S)(T_1 - t)\left(\|u_0\|^2 + \|\nabla u_0\|^2\right)
\]
\[
+ 4(1 + \beta)\left[ (R + S)(P + Q) - \left(\sqrt{PR} + \sqrt{QS}\right)^2\right]
\]
\[
\geq (-4 - 8\beta)E(0)J(t) \left(\frac{1}{\beta}\right).
\]
(4.28)

From (4.23) we obtain
\[
J'(t) \leq \beta(4 + 8\beta)E(0)J(t) \left(\frac{1}{\beta}\right).
\]
(4.29)

Now that by Lemma 2.4 that \( J(t) < 0 \) for \( t > t_0 \), multiplying (4.29) by \( J(t) \) and integrating it from \( t_0 \) to \( t \), we have
\[
J(t) \geq \omega + \kappa J(t) \left(\frac{1}{\beta}\right),
\]
(4.30)

where
\[
\omega = \beta^2 J(t_0) \left(\frac{1}{\beta}\right) \left[ Y(t_0) - \left(\|u_0\|^2 + \|\nabla u_0\|^2\right) - 8E(0)J(t_0) \left(\frac{1}{\beta}\right]\right]
\]
(4.31)

and
\[
\kappa = 8\beta^2 E(0).
\]
(4.33)
On local existence and blow-up of solution for the higher-order nonlinear Kirchhoff-type equation with nonlinear strongly damped terms

We observe that
\[
E(0) < \frac{(Y'(t_0) - ([\|u_0\|^2 + [\|\nabla u_0\|^2])}{8[Y(t_0)] + (t_0 - t_0) ([\|u_0\|^2 + [\|\nabla u_0\|^2])}
\]
\[
\omega > 0 \text{ if and only if}
\]
Then by Lemma 4.2, there exists a finite time \(T^*\), such that \(\lim_{t \to T^*} J(t) = 0\), and the upper bounds of \(T^*\) are estimated, respectively, according to the sign of \(E(0)\), we obtain
\[
\lim_{t \to T^*} (\|u(t)\|^2 + \int_0^t (\|u(s)\|^2 + [\|\nabla u(s)\|^2]) = +\infty
\]  \(4.34\)
The upper bounds of \(T^*\) are estimated as follows by Lemma 4.2
In case (1), we have
\[
T^* \leq t_0 - \frac{J(t_0)}{J(t_0)}
\]  \(4.35\)
Furthermore if
\[
J(t_0) < \min\{1, \frac{\omega}{\sqrt{\kappa}}\}
\]
then we obtain
\[
T^* \leq t_0 + \frac{1}{\sqrt{-\kappa}} Ln \frac{\omega}{\sqrt{-\kappa} - J(t_0)}
\]  \(4.36\)
In case (2),
\[
T^* \leq t_0 - \frac{J(t_0)}{J(t_0)} \quad \text{or} \quad T^* \leq t_0 - \frac{J(t_0)}{\sqrt{\omega}}
\]  \(4.37\)
In case (3),
\[
T^* \leq \frac{J(t_0)}{\sqrt{\omega}} \quad \text{or} \quad T^* \leq t_0 + 2 \frac{3^{p+1}}{2^{\beta}} \frac{\beta c}{\sqrt{\omega}} \{1 + [1 + cJ(t_0)]^{\frac{1}{2\beta}} \}
\]  \(4.38\)
Where \(c = (\kappa/\omega)^{\frac{3}{2\beta}}\), note that in case (1), \(t_0 = t^*\) is given in (4.13), and in case (2) and case (3), \(t_0 = 0\).

Remark 4.1 that in we observe that the choice of \(T^*\) in (4.21) is feasible under the same condition as in [15].

REFERENCE