International Journal of Modern Communication Technologies & Research (IJMCTR)
ISSN: 2321-0850, Volume-5, Issue-8, August 2017

Investigation of the initial-value problem for the Volterra integro-differential equations with the special structure

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***Abstract*— The solving of some problem of the natural science are reduce to solving of the initial value problem for integro-differential equations. The numerically solution of such problems has been studied relatively little. Therefore, here we consider to construction of the more exact methods and their applications to solving of the initial value problem for Volterra integro-differential equations of the second order. To this end, here it is proposed to use the second derivative multistep hybrid methods and have constructed the specific methods of the order of accuracy that were applied to solving of the model problem.**

***Index Terms*— integro-differential equations, hybrid method, initial value problem**

I. INTRODUCTION

As is known, the construction and application of the integro-differential equation with variable boundaries is connected by the name of Vito Volterra, who was investigating the linear integro-differential equation. As in classical and in some modern scientific works, are proposed to solve integro-differential equations by using the quadrature method or some its modifications. There are works in which for solving integro-differential equations of Volterra type suggest to use, spline functions, and the collocation method or finite elements.

Let us consider to solving of the following initial value problem:

$$y'' = f(x,y) + \lambda \int_{x_0}^{x} K(x,s,y(s))ds, \quad y'(x_0) = y'_0, \tag{1}$$

$$y(x_0) = y_0, \ \ x_0 \le s \le x \le X.$$

Assume that the initial value problem (1) has a continuous unique solution defined on the segment $[x_0, X]$. To find approximate values of the solution of problem (1), the segment $[x_0, X]$ is divided into N equal parts by the step size $h > 0$, and the mesh point is defined as $x_i = x_0 + ih \ (i = 0,1,..., N)$. In addition, we denote the approximate values of the solution of problem (1) by the y_m at the mesh point $x_m \ (m = 0,1,...)$ but by the $y(x_m)$ corresponding exact values of the solution of the problem (1). Note, that by the investigation of the problem (1) for the case $\lambda = 0$ and $\lambda \ne 0$ were study by many scholar (see. for example $[1]-[24]$). Here, we have used the results from the works $[1]-[9]$ and $[11]-[14]$ for the case $\lambda = 0$ which applied to solving problem (1) for the case $\lambda \ne 0$.

II. ON THE CONSTRUCTION OF THE HYBRID METHODS

As is well known, the studies of some practical problems are reduced to the initial value problem for the Volterra integro-differential equation of the second order in the case when the right hand side of the integro-differential equation is independent from the first derivative of the solution of the considering problem. For the investigation of this problem, here constructed the new method and show that the methods, specially adapted for the solving of the problem of type (1) are more effective than known. To this end, let us consider the Strorem's method, which can we written as:

$$\sum_{i=0}^{k} \alpha_i y_{n+i} = h^2 \sum_{i=0}^{k} \beta_i y''_{n+i} \, . \quad (n = 0,1,2,...). \tag{2}$$

It is known that from the convergence of the method (2) is consequently the next: $\rho(1) = \rho'(1) = 0$. Therefore $\lambda = 1$ is a double root of the characteristic polynomial $\rho(\lambda)$ of the method (2). This means that the method (2) cannot be stable. Therefore Dahlguist introduced the concept (here $\rho(\lambda) = \alpha_k \lambda^k + \alpha_{k-1}\lambda^{k+1} + ... + \alpha_1 \lambda + \alpha_0$) of 2-stability. The concept of the degree and stability for the method (2) determinate as follows.

Definition 1. Method (2) is stable if the roots of the polynomial $\rho(\lambda)$ lies inside the unit circle on the boundary of which has no roots except double root $\lambda = 1$.

Definition 2. Method (2) has the degree p, if the following is holds:

$$\sum_{i=0}^{k} (\alpha_i y(x + ih) - h^2 \beta_i y''(x + ih)) = O(h^{p+2}), h \to 0.$$

If the method (2) is stable and has the degree p, then it is converges and its velocity of the convergence equal to p.

Consider the application of the method (2) to solving of the problem (1). Then we receive:

$$\sum_{i=0}^{k} \alpha_i y_{n+i} = h^2 \sum_{i=0}^{k} \gamma_i f_{n+i} + h^2 \sum_{i=0}^{k} \gamma_i v_{n+i}, \tag{3}$$

and for the calculating of the value of the quantity v_m to propose here the following method:

$$\sum_{i=0}^{k} \alpha_i v_{n+i} = h \sum_{j=0}^{k} \sum_{i=0}^{j} \gamma_i ' K(x_{n+j}, x_{n+i}, x_{n+i}). \tag{4}$$

Here the function $v(x)$ has determined as the following:

$$v(x) = \int_{x_0}^{x} K(x, s, y(s))ds.$$

It is easy to determine that the application of the methods defined by the formula (3) and (4) to solving of scientific and technical problems are simpler. However, if the methods (3) and (4) are stable, the maximum values of the degrees of these methods are same and equal to $p_{\max} = k + 2$. In order to construct the effective methods for solving of the problem (1), let us defined the methods by using the relations (3) and (4), which have replaced by the following:

$$\sum_{i=0}^{k} \alpha_i y_{n+i} = h^2 \sum_{i=0}^{k} \gamma_i y''_{n+i} + h^2 \sum_{i=0}^{k} \beta_i y''_{n+i+\nu_i} \qquad (|\nu_i| < 1; i = 0,1,..., k), \quad (5)$$

$$\sum_{i=0}^{k} \alpha_i v_{n+i} = h \sum_{j=0}^{k} \sum_{i=0}^{j} (\hat{\gamma}_{i,j} K(x_{n+j}, x_{n+i}, y_{n+i}) + \hat{\beta}_{i,j} K(x_{n+j}, x_{n+\nu_i+i}, y_{n+i+\nu_i})) \quad . \quad (6)$$

The methods are defined by the relations (5) and (6) is more accurate than the known. But, in their application to solving of specific problems it is necessary to computing the values $y_{n+i+\nu_i} (i = 0,1,..., k)$ narrow their domain of applications. In some cases, this defect of the proposed method can be remove.

Note that the necessary conditions for the coefficients of the methods (5) (see for example [1-2]) can be written in the following form:

A. The values $\alpha_i, \beta_i, \gamma_i, \nu_i \ (i = 0,1,..., k)$ - are some real numbers, and $\alpha_k \neq 0$.

B. polynomials

$$\rho(\lambda) \equiv \sum_{i=0}^{k} \alpha_i \lambda^i; \quad \sigma(\lambda) \equiv \sum_{i=0}^{k} \beta_i \lambda^i;$$

$$\gamma(\lambda) \equiv \sum_{i=0}^{k} \gamma_i \lambda^{i+\nu_i};$$

have no common factors deferent from the constant.

C holds: $\sigma(1) + \gamma(1) \neq 0$ and $p \geq 1$.

We show that using the proposed method to solving of the problem (1) one can be construct a method with the degree $p \geq 2k + 2$.

Note that if the following method

$$\sum_{i=0}^{k} \alpha_i y_{n+i} = h^m \sum_{i=0}^{k} \beta_i y^{(m)}_{n+i}$$

applied to the solving of the following problem:

$$y^{(m)}_{(x)} = f(x, y), y^{(j)}_{(x_0)} = y_0^{(j)} (j = 0,1,..., m-1),$$

then between the order of the differential equations m and the order of the difference method k must hold the inequality $k \geq m$ (see. eg. [12]) Therefore, in the method (2) suppose that $k = 2$ and consider to the construction of some specific methods. It is known that the maximum value of the stable methods of type (2) can be obtained by the equalities $p_{\max} = k + 2$ if k is even $(k = 2\nu)$, and $p_{\max} = k + 1$ if k is odd $(k = 2\nu - 1)$. Consequently, $k = 2$ maximum order of the accuracy for the method obtained from the formula (2) is $p = 4$, which can be written as the follows:

$$y_{n+2} - 2y_{n+1} + y_n = h^2 (y''_{n+2} + 10y''_{n+1} + y''_n)/12 \quad (7)$$

This method is known as the method of Numerov (see e.g. [13, p.443]).

Obviously, for $k = 3$ a stable method received from the method (2) has a maximum degree $p = 4$ which is coincides with the method (7) (see for example [13, p.443]). Indeed, when $k = 3$ the characteristic polynomial of the stable method received from (2) has a double root $\lambda = 1$ and the root $\lambda = q$ (where $|q| < 1$). In this case, the coefficients $\alpha_i \ (i = 0,1,2,3)$ are defined as:

$$\alpha_3 = 1; \alpha_2 = q - 2; \ \alpha_1 = 1 - 2q \text{ and } \alpha_0 = q .$$

Obviously that, don't of generality, we can assume that $\beta_3 + \beta_2 + \beta_1 + \beta_0 \neq 1$ then the equation (2) can be multiplied to c^{-1}, and in the result received the polynomial $v(\lambda)$, for which will be holds $\beta_0 + \beta_1 + \beta_2 + \beta_3 = 1$. Then from the conditions $\beta_3 + \beta_2 + \beta_1 + \beta_0 = 1 + q$ consequence that $q = 0$. In this case, the system of linear algebraic equations for determining the coefficients $\beta_j \ (0 \leq j \leq 3)$ can be rewritten in the following form:

$$\beta_3 + \beta_2 + \beta_1 + \beta_0 = 1;$$
$$18\beta_3 + 12\beta_2 + 6\beta_1 = 12;$$
$$108\beta_3 + 48\beta_2 + 12\beta_1 = 50;$$
$$540\beta_3 + 160\beta_2 + 20\beta_1 = 180.$$

By solving of this system, we find that the unique solution of this system which can be written as follows:

$$\beta_3 = \beta_1 = 1/2 \ , \ \beta_3 = 10/12 \text{ and } \beta_0 = 0.$$

It is easy to verify that this method coincides with the method (7). Let us note that there is no such a fact for the follows method:

$$\sum_{i=0}^{k} a_i y_{n+i} = h \sum_{i=0}^{k} \beta_i y'_{n+i} \ .$$

Note, that for $k = 2$ from this method should be received the method of Simpson, which is stable and has the degree $p = 4$. But for $k = 3$ from the above mentioned method in particular can be obtained of the stable methods with the degree $p = 4$. For example, the following:

$$y_{m+3} - y_{m+2} = \frac{h}{24}[9y'_{m+3} + 19y'_{m+2} - 5y'_{m+1} + y'_m].$$

It follows that the mechanical generalization is not always takes place. Consequently, each of the above-proposed methods is an independent object for investigation.

It is therefore proposed here to use the following method to requires an independent approach:

$$\sum_{i=0}^{k} \alpha_i' y_{n+i} = h^2 \sum_{i=0}^{k} \beta_i' y''_{n+i+\nu_i} (|\nu_i| < 1; i = 0,1,..., k). \quad (8)$$

It can be shown that for the value $k = 2$ from the (8) one can receive the following stable methods

$$y_{n+2} = 2y_{n+1} - y_n + h^2(5y''_{n+1-\alpha} + 14y''_{n+1} + 5y''_{n+1+\alpha})/24, \quad (\alpha = \sqrt{10}/5), \tag{9}$$

$$y_{n+2} = 2y_{n+1} - y_n + h^2(4y''_{n+1-\beta} + y''_{n+1} + 4y''_{n+1+\beta})/9, \quad (\beta = \sqrt{3}/4) \tag{10}$$

Note that the method (9) has the degree $p = 6$, but the method (10) has the degree $p = 4$. In order to use the method (9) or (10), we construct some of the methods for calculating of the values the $y_{n+1\pm\alpha}$ or values $y_{n+1\pm\beta}$. Here, for the finding of the mentioned value proposed the following method:

$$y_{n+1\pm\beta} = y_{n+1} \pm \frac{\sqrt{3}}{4}(y_{n+1} - y_n) + \frac{36 \pm 7\sqrt{3}}{3\times128}h^2 y''_{n+1} \pm \frac{61\sqrt{3}}{3\times128}h^2 y''_n. \tag{11}$$

Local error for the methods of (11) is $O(h^4)$ which is consistent according to the accuracy of the method (9).

It is clear that for the above proposed methods must be known the starting values y_0 and y_1. To take into account that the y_0 - is known from the original problem, because to finding of the value y_1, here proposed the following scheme:

$$y'_{1/3} = y'_0 + \frac{h}{3}y''_0 + \frac{h^2}{18}y'''_0,$$

$$\hat{y}'_1 = y'_0 + hy''_0 + \frac{h^2}{2}y'''_0,$$

$$y_1 = y_0 + h(3y'_{1/3} + y'_1)/4.$$

Now consider the application of the hybrid methods to solving of the problem (1). For this purpose, we write that in the form:

$$y_{n+2} = 2y_{n+1} - y_n + h^2(4f_{n+1-\beta} + f_{n+1} + 4f_{n+1+\beta})/9 + h^2(4v_{n+1-\beta} + v_{n+1} + 4v_{n+1+\beta})/9, (\beta = \sqrt{3}/4).$$

And it is possible to use the Simson's method for calculating quantity v_m. Then we have:

$$v_{n+1} = v_n + h(2K(x_{n+1}, x_{n+1/3}, y_{n+1/3}) + K(x_{n+1/3}, x_{n+1/3}, y_{n+1/3}) + K(x_{n+1}, x_{n+1}, y_{n+1}))/4, \tag{12}$$

$$v_{n+v} = v_n + vh(K(x_n, x_n, y_n) + K(x_{n+1}, x_n, y_n) + 2K(x_{n+v}, x_{n+v}, y_{n+v}))/4, \quad v = 1+\beta \tag{13}$$

The following method can be used for finding of the values of $y_{n+1/3}$:

$$y_{n+1-\gamma} = (1-\gamma)y_{n+1} + \gamma y_n + \gamma h^2((\gamma^2 - 3\gamma + 2)y''_{n+1} + (1-\gamma^2)y''_n)/6. \tag{14}$$

Note that methods (12) and (13) are implicit. Therefore, for their application to solving of specific problems, one can use the predictor and corrector methods. In this case one of the following methods (see eg. [13, p.442]) can be used as the predictor :

$$y_{n+2} = 2y_{n+1} - y_n + h^2 y''_{n+1},$$

$$y_{n+2} = 2y_{n+1} - y_n + h^2(13y''_{n+1} - 2y''_n + y''_{n-1})/12.$$

To illustrate the results received here consider the finding of the solution of the following problem.

Example 1. Consider to the application of Numerov's method to solving the next problem:

$$y'' = \lambda^2 + \lambda^3 \int_o^x y(s)ds; \quad y(0) = 1; \quad y'(0) = \lambda; \; x \in [0,1],$$

(exact solution $y(x) = \exp(\lambda x)$)

by using the method of (7), which is implicit. Therefore, to use the method (7) we offer the following scheme of predictor-corrector method:

$$\hat{y}_{n+2} = 2y_{n+1} - y_n + h^2 f_{n+1} + h^2 v_{n+1},$$

$$v_{n+2} = v_n + h(K(x_{n+2}, x_{n+2}, y_{n+2}) + 4K(x_{n+1}, x_{n+1}, y_{n+1}) + K(x_n, x_n, y_n))/3,$$

$$y_{n+2} = 2y_{n+1} - y_n + h^2(\hat{f}_{n+2} + 10f_{n+1} + f_n)/12 + h^2(v_{n+2} + 10v_{n+1} + v_n)/12.$$

The problem have solved for the values $\lambda = \pm1$ and $\lambda = \pm5$ to the steps size $h = 0{,}1$; $h = 0{,}05$ and $h = 0{,}01$. Some of the received results are located in the following tables:

Table 1. The error in $y(x)$ of the method (7) for the $\lambda = 1$

Variable x	Step size $h = 0.1$	Step size $h = 0.05$	Step size $h = 0.01$
0.20	4.45-09	3.8E-10	7.32-13
0.60	5.29-08	3.45E-09	5.68-12
1.00	1.2E-07	7.51E-09	1.2E-11

Table 2. The error in $y(x)$ of the method (7) for the $h = 0{,}01$

Step size	Variable x	$\lambda = -1$	$\lambda = 5$	$\lambda = -5$
$h = 0{,}01$	0.20	8.34E-13	7.51E-09	1.49E-08
	0.60	8.56E-12	1.68E-07	6.29E-08
	1.00	2.55E-11	3.82E-06	1.35E-07

III. Conclusion

Note that the hybrid method (10) and the method of Numerov are stable and have the same order of the accuracy. In the constructing of this algorithm for applying it to solving of some problems, here have been used the predictor methods with the same degrees. Therefore in the above mentioned table have used the results corresponding to different cases for the values of the quantity λ for the Numerov's method, because results received by the method of (10) are correspond to results received by the method (7).

Here are constructed some specific hybrid methods, for using simple algorithms and also are constructed the methods to the computation of the values of the solution of the original problem in hybrid points. Remark, that for solving initial-value problem for the Volterra integro-differential equation of the second order with the special structure, here have constructed effective methods having the degree $p = 4$ and $p = 6$ which are illustrated

ACKNOWLEDGMENT

The authors express their thanks to the academician Ali Abbasov for his suggestion to investigate the computational aspects of our problem. This work was supported by the Science Development Foundation under the President of the Republic of Azerbaijan – Grant № EIF-KETPL-2-20151(25)-56/07/1.

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