The pullback attractors for the Higher-order Kirchhoff-type equation with strong linear damping

Guoguang Lin, Yunlong Gao

***Abstract*— The paper investigates pullback the attractors for the Higher-order Kirchhoff-type equation with strong linear damping:**

$$\frac{\partial^2 u}{\partial t^2}+(-\Delta)^m\frac{\partial u}{\partial t}+\left(\alpha+\beta\left\|\nabla^m u\right\|^2\right)^q(-\Delta)^m u+g(u)=f(x)+h(t,u_t).$$

Firstly, we do priori estimation for the equations to obtain the existence and uniqueness of the solution in $u\in C^0([\tau-r,\infty);V)\bigcap C^1([\tau-r,\infty);H)$ by some assumptions the Galerkin method. Then, we prove existence of the pullback attractors $\{\mathscr{A}(t)\}_{t\in R}$ in $u\in C^0([\tau-r,\infty);V)\bigcap C^1([\tau-r,\infty);H)$.

***Index Terms*— Nonlinear Higher-order Kirchhoff type equation, Galerkin method, The existence and uniqueness, The Pullback attractors**

hematics Classification:35B41, 35G31

I. INTRODUCTION

In this paper, we are concerned with the existence of pullback attractors for the following nonlinear Higher-order Kirchhoff-type equations:

$$\begin{aligned}&\frac{\partial^2 u}{\partial t^2}+(-\Delta)^m\frac{\partial u}{\partial t}+\left(\alpha+\beta\left\|\nabla^m u\right\|^2\right)^q(-\Delta)^m u+g(u)=f(x)+h(t,u_t),t>\tau,\\&u(x,t)=\phi(x,t-\tau),x\in\Omega,t\in[\tau-r,\tau],\\&\frac{\partial u}{\partial t}(x,t)=\frac{\partial\phi}{\partial t}(x,t-\tau),x\in\Omega,t\in[\tau-r,\tau],\\&u(x,t)=0,\frac{\partial^i u}{\partial v^i}=0,i=1,\ldots,m-1,x\in\partial\Omega,t\in[\tau-r,+\infty),\end{aligned} \tag{1.1}$$

where $m>1$ is an integer constant, $\alpha>0,\beta>0$ are constants and q is a real number, ϕ is the initial datum on the interval $[\tau-r,\tau]$ where $r>0$. Moreover, Ω is a bounded domain in R^n with the smooth boundary $\partial\Omega$ and v is the unit outward normal on $\partial\Omega$. $g(u)$ is a nonlinear function specified later, and u_t is defined for $\theta\in[-r,0]$ as $u_t(\theta)=u(t+\theta)$.

It is known that Kirchhoff [1] first investigated the following nonlinear vibration of an elastic string for $\delta=f=0$:

$$\rho h\frac{\partial^2 u}{\partial t^2}+\delta\frac{\partial u}{\partial t}=\left\{p_0+\frac{Eh}{2L}\int_0^L\left(\frac{\partial u}{\partial x}\right)^2dx\right\}\frac{\partial^2 u}{\partial x^2}+f;\quad 0\le xL,t\ge 0, \tag{1.2}$$

where $u=u(x,t)$ is the lateral displacement at the space coordinate x and the time t, ρ the mass density, h the cross-section area, L the length, E the Young modulus, p_0 the initial axial tension, δ the resistance modulus, and f the external force.

In [2], the existence of a pullback and forward attractors is proved for a damped wave equation with delays:

$$\begin{aligned}&\frac{\partial^2 u}{\partial t^2}+\alpha\frac{\partial u}{\partial t}-\Delta u=f+h(t,u_t),t>\tau,\\&u|_\Gamma=0,t\ge\tau-r,\\&u(x,t)=\phi(x,t-\tau),x\in\Omega,t\in[\tau-r,\tau],\\&\frac{\partial u}{\partial t}(x,t)=\frac{\partial\phi}{\partial t}(x,t-\tau),x\in\Omega,t\in[\tau-r,\tau],\end{aligned} \tag{1.3}$$

where $\Omega\subset R^n,n\ge 1$, be an open and bounded subset with smooth boundary $\partial\Omega=\Gamma$. $f+h(t,u_t)$ is the source intensity which may depend on the history of the solution, α is a positive constant, φ is the initial datum on the interval $[\tau-r,\tau]$ where $r>0$, and u_t is defined for $\theta\in[-r,0]$ as $u_t(\theta)=u(t+\theta)$.

In [3], Guoguang Lin, Fangfang Xia and Guigui Xu had studied the global and pullback attractors for a strongly damped wave equation with delays when the force term belongs to different space:

$$\frac{\partial^2 u}{\partial t^2}+\alpha\frac{\partial u}{\partial t}-\beta\Delta\frac{\partial u}{\partial t}-\Delta u+g(u)=f(x)+h(t,u_t),t>\tau. \tag{1.4}$$

In [4], authors consider non-autonomous dynamical behavior of wave-type evolutionary, on a bounded domain Ω in R^3, with smooth boundary $\partial\Omega$:

$$\begin{aligned}&u_{tt}+h(u_t)-\Delta u+f(u,t)=g(x,t),x\in\Omega,\\&u|_{\partial\Omega}=0,x\in\partial\Omega,\\&u(x,0)=u_0(x),u_t(x,0)=v_0(x),x\in\partial\Omega,\end{aligned} \tag{1.5}$$

where $g(x,t)\in L^2_{loc}(R;L^2(\Omega))$,and $h(u_t),f:R\to R$ and verify some of assumptions.

Authors establish a criterion for the existence of pullback attractors. Moreover, they show that the pullback

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k-contraction is not equivalent to the pullback asymptotic compactness, unless the cocycle mapping has a nested bounded pullback absorbing set.

In [5], authors study existence of pullback attractors for the following functional Navier-Stokes problem:

$$\frac{\partial u}{\partial t}-\nu\Delta u+\sum_{i=1}^{2}u_i\frac{\partial u}{\partial x_i}=f(t,u(t-\rho(t)))-\nabla p+g(t),(x,t)\in(\tau,+\infty)\times\Omega,$$
$$divu=0,(x,t)\in(\tau,+\infty)\times\Omega,$$
$$u=0,(x,t)\in(\tau,+\infty)\times\Gamma,$$
$$u(\tau+t,x)=\phi(t,x),t\in[-h,0],x\in\Omega, \quad (1.6)$$

where $\Omega\subset R^2$ is an open bounded set with regular boundary $\Gamma,\upsilon>0$ is the kinemtic viscosity, u is the velocity field of the fluid, p is the pressure, g,f are external force term, ρ is an adequate given delay function.

Authors prove the existence of a unique pullback attractor in higher regularity space for the multivalued process associated with the nonautonomous 2D-Navier-Stokes model with delays and without the uniqueness of solutions.

Some people have studied for equations of the form:

$$u'+A(t)u(t)=F(t,u_t),t\ge 0,$$
$$u(t)=\psi(t),t\in[-h,0]. \quad (1.7)$$

For example, M.J.Garrido and J.Real of [5] had proved some results on the existence and uniqueness of solution for a class of evolution equations of second order in time, containing some hereditary characteristics.

At present, most people had investigated global attractors, exponential attractors and blow-up of Higher-order Kirchhoff-type equations, and we can see [6-32]. Because equations of the paper posses $g(u):R\to R$ and $h(t,u_t)$, they increase difficulties for existence of solutions. We establish pullback attractors omit [2].

In order to make these equations more normal, in section 2, some assumptions, notations and the main lemmas are given. In section 3, Under these assumptions, we prove the existence and uniqueness of solution for the problems (1.1). In section 4, we prove existence of the pullback attractor similar to [2].

II. Preliminaries

2.1 Assumptions and some of lemmas

In this section, we introduce material needed in the proof our main result. We use the standard Lebesgue space $L^p(\Omega)$ and Sobolev space $H^m(\Omega)$ with their usual scalar products and norms. Meanwhile we define:

$$H_0^m(\Omega)=\left\{u\in H^m(\Omega):\frac{\partial^i u}{\partial v^i}=0,i=0,1,\dots,m-1\right\},$$

and introduce the following abbreviations:

$$E_0=H_0^m(\Omega)\times L^2(\Omega),E_1=H^{2m}(\Omega)\cap H_0^m(\Omega)\times H_0^m(\Omega),H=L^2(\Omega),V=H_0^m(\Omega),$$
$$A=-\Delta,\|\cdot\|_{H^m}=\|\cdot\|_{H^m(\Omega)},\|\cdot\|_{H_0^m}=\|\cdot\|_{H_0^m(\Omega)},\|\cdot\|=\|\cdot\|_{L^2(\Omega)},\|\cdot\|_p=\|\cdot\|_{L^p(\Omega)}$$

for any real number $p>1$, and λ_1 is the first eigenvalue of A.

(1.1) can be written as a second order differential equation in H:

$$u''+(-\Delta)^m u'+\left(\alpha+\beta\|\nabla^m u\|^2\right)^q(-\Delta)^m u+g(u)=f(x)+h(t,u_t),t>\tau,$$
$$u(t)=\phi(t-\tau),t\in[\tau-r],$$
$$u'(t)=\phi'(t-\tau),t\in[\tau-r]. \quad (2.1)$$

In general, if $(X,\|\cdot\|_X)$ is a Banach space, we denote by C_X the space $C^0([-r,0];X)$ with the sup-norm, i.e. $\|\phi\|_{C_X}=\sup_{\theta\in[-r,0]}\|\phi(\theta)\|_X$, for $\phi\in C_X$. Given another Banach space $(Y,\|\bullet\|_Y)$ such that the injection $X\subset Y$ is continuous, we denote by $C_{X,Y}$ the Banach space $C_X\cap C^1([-r,0];Y)$ with the norm $\|\bullet\|_{C_{X,Y}}$ defined by:

$$\|\phi\|_{C_{X,Y}}^2=\|\phi\|_{C_X}^2+\|\phi_t\|_{C_Y}^2 \quad ,\text{for}\quad \phi\in C_{X,Y}. \quad (2.2)$$

According to [2] and [8], we present some assumptions and notations needed in the proof of our results. For this reason, we assume nonlinear term $g(u)\in C^1(\Omega)$ satisfies that:

(H_1) Setting $G(s)=\int_0^s g(r)dr$, then

$$\liminf_{|s|\to\infty}\frac{G(s)}{s^2}\ge 0; \quad (2.3)$$

(H_2) If

$$\limsup_{|s|\to\infty}\frac{|g'(s)|}{|s|^r}=0, \quad (2.4)$$

where $0\le r<+\infty(n\le 2m),0\le r<2(2m<n\le 2m+1),r=0(n\ge 2m+2).$

(H_3) There exist constant $C_0>0$, such that

$$\liminf_{|s|\to\infty}\frac{sg(s)-C_0G(s)}{s^2}\ge 0. \quad (2.5)$$

(H_4) There exist constant $C_1>0$, such that

$$|g(s)|\le C_1\left(1+|s|^p\right), \quad (2.6)$$
$$|g'(s)|\le C_1\left(1+|s|^{p-1}\right), \quad (2.7)$$

where $1\le p\le\frac{n}{n-2m}(n>2m)$ and $1\le p<+\infty(n\le 2m)$.

Now, we make the following hypotheses on the function $h:R\times C_H\to H$:

(G_1) $\forall\xi\in C_H,t\in R\to h(t,\xi)\in H$ is continuous;

(G_2) $\forall t\in R,h(t,0)=0$;

(G_3) $\exists L_h>0$ such that $\forall t\in R,\forall\xi,\eta\in C_H$,

$$\|h(t,\xi)-h(t,\eta)\|\le L_h\|\xi-\eta\|_{C_H}; \quad (2.8)$$

(G_4) $\exists k_0 \geq 0, C_h > 0$ such that $\forall k \in [0, k_0], \tau \leq t, u, v \in C^0([\tau - r, t]; H)$,

$$\int_\tau^t e^{ks} \|h(s,u_s) - h(s,v_s)\|^2 ds \leq C_h^2 \int_{\tau-r}^t e^{ks} \|u(s) - v(s)\|^2 ds. \quad (2.9)$$

For every $\gamma > 0$, by $(H_1)-(H_3)$ and apply Poincare inequality, there exist constants $C(\gamma) > 0$, such that

$$J(u) + \gamma \|\nabla^m u\|^2 + C(\gamma) \geq 0, \forall u \in H^m(\Omega), \quad (2.10)$$

$$(g(u), u) - C_2 J(u) + \gamma \|\nabla^m u\|^2 + C(\gamma) \geq 0, \forall u \in H^m(\Omega), \quad (2.11)$$

where $J(u) = \int_\Omega G(u)dx, 0 < C_2 < \frac{3q}{4} + \frac{3}{4}$ is independent of γ.

Lemma 2.1. (Young's Inequality)(See [26]) For any $\varepsilon > 0$ and $a, b \geq 0$, then

$$ab \leq \frac{\varepsilon^p}{p} a^p + \frac{1}{q\varepsilon^q}, \quad (2.12)$$

where $\frac{1}{p} + \frac{1}{q} = 1, p > 1, q > 1$.

Lemma 2.2. (Sobolev-Poincare inequality)(See [20]) Let s be a number with

$2 \leq s < +\infty, n \leq 2m$ and $2 \leq s \leq \frac{2m}{n-2m}, n > 2m$.

Then there is a constant k depending on Ω and s such that

$$\|u\|_s \leq K \left\|(-\Delta)^{\frac{m}{2}} u\right\|, \forall u \in H_0^m(\Omega). \quad (2.13)$$

Lemma 2.3.(Gronwall's inequality)(See [26]) If $\forall t \in [t_0, +\infty), y(t) \geq 0$ and $\frac{dy}{dt} + gy \leq h$, such that

$$y(t) \leq y(t_0) e^{-g(t-t_0)} + \frac{h}{g}, t \geq t_0, \quad (2.14)$$

where $g > 0, h \geq 0$ are constants.

Lemma 2.4. (See [7]) Let ψ be an absolutely continuous positive function on R^+, which satisfies for some $\grave{o} > 0$ the differential inequality

$$\frac{d}{dt}\psi(t) + 2\grave{o}\psi(t) \leq g(t)\psi(t) + h(t), t > 0, \quad (2.15)$$

where $h \in L^1_{loc}(R^+)$ and

$$\int_\tau^t g(\tau)d\tau \leq \grave{o}(t-\tau) + m, \text{ for } t \geq \tau \geq 0, \quad (2.16)$$

with some $m > 0$. Then

$$\psi(t) \leq e^m \left(\psi(s) e^{-\grave{o}(t-s)} + \int_s^t |h(\tau)| e^{-\grave{o}(t-y)} dy \right), \forall t \geq s \geq 0. \quad (2.17)$$

2.2 Preliminaries on pullback attractors

We deal with the global attractors by semigroup $S(t)$. Instead of a family of the one-parameter semigroup or process $U(t,\tau)$ on the complete metric space $X, U(t,\tau)\psi$ denotes the solution at time t which was equal to the initial value ψ at time τ.

The semigroup property is replaced by process composition property:

$$U(t,\tau)U(\tau,r) = U(t,r), \text{ for all } t \geq \tau \geq r, \quad (2.18)$$

and obviously, the initial condition implies $U(\tau,\tau) = Id$.

Definition 2.1. (See [2]) Let U be the two-parameter semigroup or process on the complete metric space X. A family of compact set $A(t)_{t \in R}$ is said to be a pullback attractor for U ,if for all $\tau \in R$. It satisfies:(1) $U(t,\tau)A(\tau) = A(\tau)$,for all $t \geq \tau$; (2) $\lim_{s \to \infty} dist_X(U(t,t-s)D, A(t)) = 0$ for all bounded $D \subset X$, and all $t \in R$.

Definition 2.2. (See [2]) If the family $B(t)_{t \in R}$ satisfying:

(1) pullback absorbing with respect to the process U ,if for all $t \in R$ and all bounded $D \subset X$,there exists $T_D > 0$ such that $U(t,t-s)D \subset B(t)$ for all $s > T_D(t)$;

(2) pullback attracting with respect to the process U, if for all $t \in R$, all bounded $D \subset X$, and all $\varepsilon > 0$, there exists $T_{\varepsilon,D}(t) > 0$ such that for all $s > T_{\varepsilon,D}(t)$,

$$dist_X(U(t,t-s)D, B(t)) < \varepsilon; \quad (2.19)$$

(3) pullback uniformly absorbing (respectively uniformly attracting) if $T_D(t)$ in pact (1) (respectively $T_{\varepsilon,D}(t) > 0$ in part (2)) does not depend on time t.

Theorem 2.2. (See [2]) Let $U(t,\tau)$ be a two-parameter process, and suppose $U(t,\tau): X \to X$ is continuous for all $t \geq \tau$. If there exists a family of compact pullback attracting sets $B(t)_{t \in R}$, then there exists a pullback attractor $A(t)_{t \in R}$, such that $A(t) \subset B(t)$ for all $t \in R$, and which is given by

$$A = \overline{\bigcup_{D \subset X} \Lambda_D(t)}$$ where

$$\Lambda_D(t) = \bigcap_{n \in N} \overline{\bigcup_{s \geq n} U(t-s)D}. \quad (2.20)$$

III. EXISTENCE AND UNIQUENESS OF THE SOLUTION

Lemma 3.1. Assume that $f \in H, \phi \in C_{V,H}$ and $g(u)$ satisfies $(H_1)-(H_3)$, h satisfies $(G_1)-(G_5)$, and

$$\left(\frac{2C_h^2}{C_2}\right)^{\frac{1}{4}} < \min\left\{2, \frac{q\lambda_1^m}{2}, \frac{\beta^q\lambda_1^m}{2}, \frac{-2-C_2-\lambda_1^m+\sqrt{(2+C_2+\lambda_1^m)^2+16\lambda_1^m}}{4}\right\}. \tag{3.1}$$

Then, for any $\tau \in R$, there exists a unique solution $u(\cdot)=u(\cdot;\tau,\phi)$ of the problem (1.1)-(1.4) and $C_3>0$, such that

$$u \in C^0([\tau-r,\infty);V)\bigcap C^1([\tau-r,\infty);H), \tag{3.2}$$

and

$$\int_t^\infty \left\|\nabla^m u_t(s)\right\|^2 ds \le C_3, t\ge\tau. \tag{3.3}$$

Proof. Step1: existence of the solution

We take the scalar product in L^2 of equation (1.1) with $v=u'+\varepsilon u$ and

$$\left(\frac{2C_h^2}{C_2}\right)^{\frac{1}{4}} < \varepsilon < \min\left\{2, \frac{q\lambda_1^m}{2}, \frac{\beta^q\lambda_1^m}{2}, \frac{-2-C_2-\lambda_1^m+\sqrt{(2+C_2+\lambda_1^m)^2+16\lambda_1^m}}{4}\right\}.$$

Then

$$\left(u''+(-\Delta)^m u'+\left(\alpha+\beta\left\|\nabla^m u\right\|^2\right)^q(-\Delta)^m u+g(u),v\right)=(f(x),v). \tag{3.4}$$

By using Poincare's inequality and Young's inequality, after a computation in (3.4), we have

$$\begin{aligned}(u'',v)&=(v'-\varepsilon u',v)\\&=\frac{1}{2}\frac{d}{dt}\|v\|^2-\varepsilon(v-\varepsilon u,v)\\&=\frac{1}{2}\frac{d}{dt}\|v\|^2-\varepsilon\|v\|^2+\varepsilon^2(u,v)\\&\ge\frac{1}{2}\frac{d}{dt}\|v\|^2-\left(\varepsilon+\frac{\varepsilon^2}{2}\right)\|v\|^2-\frac{\varepsilon^2}{2\lambda_1^m}\left\|\nabla^m u\right\|^2.\end{aligned} \tag{3.5}$$

$$\begin{aligned}\left((-\Delta)^m u',v\right)&=\left((-\Delta)^m v-\varepsilon(-\Delta)^m u,v\right)\\&=\left\|\nabla^m v\right\|^2-\varepsilon\left(\nabla^m u,\nabla^m v\right)\\&\ge\left(1-\frac{\varepsilon}{2}\right)\left\|\nabla^m v\right\|^2-\frac{\varepsilon}{2}\left\|\nabla^m u\right\|^2\\&\ge\left(\lambda_1^m-\frac{\lambda_1^m\varepsilon}{2}\right)\|v\|^2-\frac{\varepsilon}{2}\left\|\nabla^m u\right\|^2,\end{aligned} \tag{3.6}$$

with $0<\varepsilon<2$.

$$\begin{aligned}&\left(\left(\alpha+\beta\left\|\nabla^m u\right\|^2\right)^q(-\Delta)^m u,v\right)\\&=\frac{1}{2}\left(\alpha+\beta\left\|\nabla^m u\right\|^2\right)^q\frac{d}{dt}\left\|\nabla^m u\right\|^2+\varepsilon\left(\alpha+\beta\left\|\nabla^m u\right\|^2\right)^q\left\|\nabla^m u\right\|^2\\&=\frac{1}{2\beta(q+1)}\frac{d}{dt}\left(\alpha+\beta\left\|\nabla^m u\right\|^2\right)^{q+1}+\frac{\varepsilon}{\beta}\left(\alpha+\beta\left\|\nabla^m u\right\|^2\right)^{q+1}-\frac{\alpha\varepsilon}{\beta}\left(\alpha+\beta\left\|\nabla^m u\right\|^2\right)^q.\end{aligned} \tag{3.7}$$

$$(g(u),v)=\frac{d}{dt}J(u)+\varepsilon(g(u),u). \tag{3.8}$$

$$\left(f(x),v\right)+\left(h(t,u_t),v\right)\le\frac{\|f\|^2}{\varepsilon^2}+\frac{\|h(t,u_t)\|^2}{\varepsilon^2}+\frac{\varepsilon^2}{2}\|v\|^2. \tag{3.9}$$

Substituting (3.5)-(3.9) into (3.4), then

$$\begin{aligned}&\frac{d}{dt}\left[\|v\|^2+\frac{1}{\beta(q+1)}\left(\alpha+\beta\left\|\nabla^m u\right\|^2\right)^{q+1}+2J(u)\right]+\left[2\lambda_1^m-\left(2+\lambda_1^m\right)\varepsilon-2\varepsilon^2\right]\|v\|^2\\&+\frac{2\varepsilon}{\beta}\left(\alpha+\beta\left\|\nabla^m u\right\|^2\right)^{q+1}-\frac{2\alpha\varepsilon}{\beta}\left(\alpha+\beta\left\|\nabla^m u\right\|^2\right)^q-\left(\varepsilon+\frac{\varepsilon^2}{\lambda_1^m}\right)\left\|\nabla^m u\right\|^2+2\varepsilon(g(u),u)\\&\le\frac{2\|f\|^2}{\varepsilon^2}+\frac{2\|h(t,u_t)\|^2}{\varepsilon^2}.\end{aligned} \tag{3.10}$$

Next, some of the items are estimated in (3.10). By Young's inequality, we have

$$\left\|\nabla^m u\right\|^2\le\frac{1}{q+1}\left\|\nabla^m u\right\|^{2q+2}+\frac{q}{q+1}, \tag{3.11}$$

$$\left\|\nabla^m u\right\|^2\le\frac{\beta^q}{4(q+1)}\left\|\nabla^m u\right\|^{2q+2}+\frac{q\left(\frac{4}{\beta^q}\right)^{\frac{1}{q}}}{q+1}, \tag{3.12}$$

$$\left(\alpha+\beta\left\|\nabla^m u\right\|^2\right)^q\le\frac{q}{2\alpha(q+1)}\left(\alpha+\beta\left\|\nabla^m u\right\|^2\right)^{q+1}+\frac{(2\alpha)^q}{q+1}. \tag{3.13}$$

By (2.10)-(2.11), (3.11)-(3.13), we have

$$\begin{aligned}&\frac{1}{\beta(q+1)}\left(\alpha+\beta\left\|\nabla^m u\right\|^2\right)^{q+1}+2J(u)+C(\gamma)+\frac{\beta^q q}{q+1}\\&\ge\frac{\beta^q}{q+1}\left\|\nabla^m u\right\|^{2q+2}+2J(u)+C(\gamma)+\frac{\beta^q q}{q+1}\\&\ge\beta^q\left\|\nabla^m u\right\|^2+2J(u)+2C(\gamma)\\&\ge\frac{2\varepsilon}{\lambda_1^m}\left\|\nabla^m u\right\|^2+2J(u)+2C(\gamma)\\&\ge\frac{\varepsilon}{\lambda_1^m}\left\|\nabla^m u\right\|^2\\&\ge 0,\end{aligned} \tag{3.14}$$

$$\begin{aligned}
&\frac{2\varepsilon}{\beta}\left(\alpha+\beta\|\nabla^m u\|^2\right)^{q+1}-\frac{2\alpha\varepsilon}{\beta}\left(\alpha+\beta\|\nabla^m u\|^2\right)^{q}-\left(\varepsilon+\frac{\varepsilon^2}{\lambda_1^m}\right)\|\nabla^m u\|^2+2\varepsilon(g(u),u)\\
&\geq\frac{\varepsilon}{\beta}\left(\alpha+\beta\|\nabla^m u\|^2\right)^{q+1}+\left[\frac{2\alpha(q+1)\varepsilon}{q\beta}-\frac{2\alpha\varepsilon}{\beta}\right]\left(\alpha+\beta\|\nabla^m u\|^2\right)^{q}-\frac{(2\alpha)^{q+1}\varepsilon}{q\beta}-\left(\varepsilon+\frac{\varepsilon^2}{\lambda_1^m}\right)\|\nabla^m u\|^2+2\varepsilon(g(u),u)\\
&\geq\frac{\varepsilon}{\beta}\left(\alpha+\beta\|\nabla^m u\|^2\right)^{q+1}-\left(\varepsilon+\frac{\varepsilon^2}{\lambda_1^m}\right)\|\nabla^m u\|^2+2\varepsilon(g(u),u)-\frac{(2\alpha)^{q+1}\varepsilon}{q\beta}\\
&\geq\frac{3\varepsilon}{4\beta}\left(\alpha+\beta\|\nabla^m u\|^2\right)^{q+1}+\frac{\beta^q\varepsilon}{4}\|\nabla^m u\|^{2q+2}-\left(\varepsilon+\frac{2\varepsilon^2}{\lambda_1^m}\right)\|\nabla^m u\|^2+2\varepsilon C_2J(u)-2\varepsilon C(\gamma)-\frac{(2\alpha)^{q+1}\varepsilon}{q\beta}\\
&\geq\frac{3\varepsilon}{4\beta}\left(\alpha+\beta\|\nabla^m u\|^2\right)^{q+1}+\left(q\varepsilon-\frac{2\varepsilon^2}{\lambda_1^m}\right)\|\nabla^m u\|^2+2\varepsilon C_2J(u)-2\varepsilon C(\gamma)-\frac{(2\alpha)^{q+1}\varepsilon}{q\beta}-\frac{4^{\frac{1}{q}}q\varepsilon}{\beta}\\
&\geq\frac{3\varepsilon}{4\beta}\left(\alpha+\beta\|\nabla^m u\|^2\right)^{q+1}+2\varepsilon C_2J(u)-2\varepsilon C(\gamma)-\frac{(2\alpha)^{q+1}\varepsilon}{q\beta}-\frac{4^{\frac{1}{q}}q\varepsilon}{\beta},
\end{aligned}\qquad (3.15)$$

where $0<\varepsilon<\min\left\{\frac{q\lambda_1^m}{2},\frac{\beta^q\lambda_1^m}{2}\right\},\gamma=\frac{\varepsilon}{2\lambda_1^m}$.

Since

$$0<C_2<\frac{3q}{4}+\frac{3}{4},0<\varepsilon<\frac{-2-C_2-\lambda_1^m+\sqrt{\left(2+C_2+\lambda_1^m\right)^2+16\lambda_1^m}}{4},$$

and

(3.14)-(3.15)are substituted into (3.10), then

$$\begin{aligned}
&\frac{d}{dt}\left[\|v\|^2+\frac{1}{\beta(q+1)}\left(\alpha+\beta\|\nabla^m u\|^2\right)^{q+1}+2J(u)+C(\gamma)+\frac{\beta^q q}{q+1}\right]+C_2\varepsilon\left[\|v\|^2+\frac{1}{\beta(q+1)}\left(\alpha+\beta\|\nabla^m u\|^2\right)^{q+1}+2J(u)+C(\gamma)+\frac{\beta^q q}{q+1}\right]\\
&\leq\frac{d}{dt}\left[\|v\|^2+\frac{1}{\beta(q+1)}\left(\alpha+\beta\|\nabla^m u\|^2\right)^{q+1}+2J(u)+C(\gamma)+\frac{\beta^q q}{q+1}\right]+m_0\left[\|v\|^2+\frac{1}{\beta(q+1)}\left(\alpha+\beta\|\nabla^m u\|^2\right)^{q+1}+C(\gamma)+\frac{\beta^q q}{q+1}\right]+2\varepsilon C_2J(u)\\
&\leq\frac{d}{dt}\left[\|v\|^2+\frac{1}{\beta(q+1)}\left(\alpha+\beta\|\nabla^m u\|^2\right)^{q+1}+2J(u)+C(\gamma)+\frac{\beta^q q}{q+1}\right]+\left[2\lambda_1^m-\left(2+\lambda_1^m\right)\varepsilon-2\varepsilon^2\right]\|v\|^2+\frac{3\varepsilon}{4\beta}\left(\alpha+\beta\|\nabla^m u\|^2\right)^{q+1}+2\varepsilon C_2J(u)\\
&\leq\frac{2\|f\|^2}{\varepsilon^2}+\frac{2\|h(t,u_t)\|^2}{\varepsilon^2}+2\varepsilon C(\gamma)+\frac{(2\alpha)^{q+1}\varepsilon}{q\beta}+\frac{4^{\frac{1}{q}}q\varepsilon}{\beta}+m_0\left(C(\gamma)+\frac{\beta^q q}{q+1}\right),
\end{aligned}$$

. (3.16)

with $m_0=\min\left\{2\lambda_1^m-\left(2+\lambda_1^m\right)\varepsilon-2\varepsilon^2,\frac{3(q+1)\varepsilon}{4}\right\}$.

We set

$$y(t)=\|v\|^2+\frac{1}{\beta(q+1)}\left(\alpha+\beta\|\nabla^m u\|^2\right)^{q+1}+2J(u)+C(\gamma)+\frac{\beta^q q}{q+1},$$

. (3.17)

$$\tilde{C}=2\varepsilon C(\gamma)+\frac{(2\alpha)^{q+1}\varepsilon}{q\beta}+\frac{4^{\frac{1}{q}}q\varepsilon}{\beta}+m_0\left(C(\gamma)+\frac{\beta^q q}{q+1}\right). \qquad (3.18)$$

So, from (3.16) we get

$$\frac{d}{dt}y(t)+C_2\varepsilon y(t)\leq\frac{2\|f\|^2}{\varepsilon^2}+\frac{2\|h(t,u_t)\|^2}{\varepsilon^2}+\tilde{C}.$$

. . (3.19)

As our assumption ensure that $-C_2\varepsilon+\frac{2C_h^{\,2}}{\varepsilon^3}<0$, we can then choose $k\in(0,k_0)$ small enough such that $k-C_2\varepsilon+\frac{2C_h^{\,2}}{\varepsilon^3}<0$. For this choice, we have

$$\frac{d}{dt}\left[e^{kt}y(t)\right]=ke^{kt}y(t)+e^{kt}\frac{d}{dt}y(t), \qquad (3.20)$$

$$\frac{d}{dt}\left[e^{kt}y(t)\right]\leq\left(k-C_2\varepsilon\right)e^{kt}y(t)+\frac{2}{\varepsilon^2}e^{kt}\|f\|^2+\frac{2}{\varepsilon^2}e^{kt}\|h(t,u_t)\|^2+\tilde{C}e^{kt}. \qquad (3.21)$$

For (3.21), by integrating over the interval $[\tau,t]$, we deduce

$$\begin{aligned}
e^{kt}y(t)&\leq e^{k\tau}y(\tau)+\left(k-C_2\varepsilon\right)\int_\tau^t e^{ks}y(s)ds+\frac{2\|f\|^2\left(e^{kt}-e^{k\tau}\right)}{k\varepsilon^2}+\frac{2}{\varepsilon^2}\int_\tau^t e^{ks}\|h(t,u_s)\|^2ds+\tilde{C}\int_\tau^t e^{ks}ds\\
&\leq e^{k\tau}y(\tau)+\left(k-C_2\varepsilon\right)\int_\tau^t e^{ks}y(s)ds+\frac{2\|f\|^2\left(e^{kt}-e^{k\tau}\right)}{k\varepsilon^2}+\frac{2C_h^{\,2}\lambda_1^{-m}}{\varepsilon^2}\int_{\tau-r}^t e^{ks}\|\nabla^m u(s)\|^2ds+\frac{\tilde{C}}{k}\left(e^{kt}-e^{k\tau}\right)\\
&=e^{k\tau}y(\tau)+\left(k-C_2\varepsilon\right)\int_\tau^t e^{ks}y(s)ds+\frac{2\|f\|^2\left(e^{kt}-e^{k\tau}\right)}{k\varepsilon^2}+\frac{2C_h^{\,2}\lambda_1^{-m}}{\varepsilon^2}\left(\int_{\tau-r}^\tau e^{ks}\|\nabla^m u(s)\|^2ds+\int_\tau^t e^{ks}\|\nabla^m u(s)\|^2ds\right)+\frac{\tilde{C}}{k}\left(e^{kt}-e^{k\tau}\right).
\end{aligned}\qquad (3.22)$$

By (3.14), we have

$$\frac{2C_h^{\ 2}\lambda_1^{-m}}{\varepsilon^2}\int_{\tau-r}^{t}e^{ks}\left\|\nabla^m u(s)\right\|^2 ds\le\frac{2C_h^{\ 2}}{\varepsilon^3}\int_{\tau-r}^{t}e^{ks}y(s)ds. \tag{3.23}$$

Therefore, we have

$$e^{kt}y(t)\le e^{k\tau}y(\tau)+\left(k-C_2\varepsilon\right)\int_\tau^t e^{ks}y(s)ds+\frac{2\|f\|^2\left(e^{kt}-e^{k\tau}\right)}{k\varepsilon^2}+\frac{2C_h^{\ 2}}{\varepsilon^3}\int_{\tau-r}^{t}e^{ks}y(s)ds+\frac{\tilde{C}}{k}\left(e^{kt}-e^{k\tau}\right). \tag{3.24}$$

By $\phi\in C_{V,H}$,let $D\subset C_{V,H}$ be a bounded set, i.e. there exists $d>0$ such that

$$\|\phi\|_{C_V}^2+\|\phi'+\varepsilon\phi\|_{C_H}^2\le d^2, \tag{3.25}$$

$$y\left(\phi(t)\right)\le d^2. \tag{3.26}$$

From (3.25)-(3.26) and he integral value theorem, we obtain

$$\begin{aligned}
e^{kt}y(t)&\le e^{k\tau}y(\tau)+\left(k-C_2\varepsilon+\frac{2C_h^{\ 2}}{\varepsilon^3}\right)\int_\tau^t e^{ks}y(s)ds+\frac{2\|f\|^2\left(e^{kt}-e^{k\tau}\right)}{k\varepsilon^2}+\frac{2C_h^{\ 2}}{\varepsilon^3}\int_{\tau-r}^{\tau}e^{ks}y(s)ds+\frac{\tilde{C}}{k}\left(e^{kt}-e^{k\tau}\right)\\
&\le e^{k\tau}d^2+\frac{2\|f\|^2\left(e^{kt}-e^{k\tau}\right)}{k\varepsilon^2}+\left(k-C_2\varepsilon+\frac{2C_h^{\ 2}}{\varepsilon^3}\right)\int_\tau^t e^{ks}y(s)ds+\frac{2C_h^{\ 2}r}{\varepsilon^3}e^{k\tau}d^2+\frac{\tilde{C}}{k}\left(e^{kt}-e^{k\tau}\right)\\
&=\left(1+\frac{2C_h^{\ 2}r}{\varepsilon^3}\right)e^{k\tau}d^2+\frac{2\|f\|^2\left(e^{kt}-e^{k\tau}\right)}{k\varepsilon^2}+\left(k-C_2\varepsilon+\frac{2C_h^{\ 2}}{\varepsilon^3}\right)\int_\tau^t e^{ks}y(s)ds+\frac{\tilde{C}}{k}\left(e^{kt}-e^{k\tau}\right)\\
&\le\left(1+\frac{2C_h^{\ 2}r}{\varepsilon^3}\right)e^{k\tau}d^2+\frac{2\|f\|^2\left(e^{kt}-e^{k\tau}\right)}{k\varepsilon^2}+\frac{\tilde{C}}{k}\left(e^{kt}-e^{k\tau}\right).
\end{aligned} \tag{3.27}$$

Therefore, we have

$$\|v\|^2+\frac{\varepsilon}{\lambda_1^{\ m}}\left\|\nabla^m u\right\|^2\le y(t)\le\left(1+\frac{2C_h^{\ 2}r}{\varepsilon^3}\right)d^2e^{k(\tau-t)}+\frac{2\|f\|^2\left(1-e^{k(\tau-t)}\right)}{k\varepsilon^2}+\frac{\tilde{C}}{k}\left(1-e^{k(\tau-t)}\right). \tag{3.28}$$

Further, we get

$$\left\|\nabla^m u\right\|^2+\|v\|^2\le\frac{\rho_0^{\ 2}}{2}+\hat{\rho}_0^{\ 2}d^2e^{k(\tau-t)},\forall t\ge\tau, \tag{3.29}$$

where $\rho_0^{\ 2}=\frac{2\tilde{C}\lambda_1^{\ m}}{k\varepsilon}+\frac{4\|f\|^2\lambda_1^{\ m}}{k\varepsilon^3},\hat{\rho}_0^{\ 2}=\frac{\lambda_1^{\ m}}{\varepsilon}+\frac{2C_h^{\ 2}r\lambda_1^{\ m}}{\varepsilon^4}$.

Then (3.29) yields that

$$\left\|\nabla^m u(t;\tau,\phi)\right\|^2+\left\|u'(t;\tau,\phi)\right\|^2\le\frac{\rho_0^{\ 2}}{2}+\hat{\rho}_0^{\ 2}d^2e^{k(\tau-t)},\forall t\ge\tau, \tag{3.30}$$

and, in particular,

$$\left\|\nabla^m u(t;\tau,\phi)\right\|^2+\left\|u'(t;\tau,\phi)\right\|^2\le\frac{\rho_0^{\ 2}}{2}+\hat{\rho}_0^{\ 2}d^2,\forall t\ge\tau, \tag{3.31}$$

Moreover, as $u(t;\tau,\phi)=\phi(t-\tau)$ and $u'(t;\tau,\phi)=\phi'(t-\tau)$ for $t\in[\tau-r,\tau]$, then (3.30) holds true for $t\ge\tau-r$.

By Galerkin method, we get $u\in C^0([\tau-r,\infty);V)\cap C^1([\tau-r,\infty);H)$.

Step 2: uniqueness of the solution

Assume that $u(\cdot)=u(\cdot;\tau,\phi)$ and $v(\cdot)=v(\cdot;\tau,\psi)$ are two solutions of the initial boundary value problem (1.1), ϕ,ψ are the corresponding initial value, we denote $w(\cdot)=u(\cdot)-v(\cdot)$. Therefore we have

$$\begin{aligned}
&w''+\left(-\Delta\right)^m w'+M(t)\left(-\Delta\right)^m w+\bar{M}(t)\left(\nabla^m(u+v),\nabla^m w\right)\left(-\Delta\right)^m(u+v)+g(u)-g(v)=h(t,u_t)-h(t,v_t),t\ge\tau,\\
&w(t)=\phi(t-\tau)-\psi(t-\tau),t\in[\tau-r,\tau],\\
&w'(t)=\phi'(t-\tau)-\psi'(t-\tau),t\in[\tau-r,\tau],
\end{aligned} \tag{3.32}$$

where

$$M(t)=\frac{1}{2}\left[\left(\alpha+\beta\left\|\nabla^m u\right\|^2\right)^q+\left(\alpha+\beta\left\|\nabla^m v\right\|^2\right)^q\right]\ge\alpha^q, \tag{3.33}$$

$$\bar{M}(t)=\frac{1}{2}\int_0^1 q\beta\left[\alpha+\beta\left(\lambda\left\|\nabla^m u\right\|^2+(1-\lambda)\left\|\nabla^m v\right\|^2\right)\right]^{q-1}d\lambda\ge 0. \tag{3.34}$$

Using the multiplier $w' + \varepsilon w$ in (3.32), we have

$$\frac{d}{dt}H(t) + \left\|\nabla^m w'\right\|^2 = K(t) - \left(g(u) - g(v), w' + \varepsilon w\right) + \left(h(t,u_t) - h(t,v_t), w' + \varepsilon w\right), \tag{3.35}$$

with

$$H(t) = \frac{1}{2}\left(\left\|w'\right\|^2 + \left\|\nabla^m w\right\|^2\right) + \varepsilon\left(w, w'\right), \tag{3.36}$$

$$\begin{aligned} K(t) = & -M(t)(\nabla^m w, \nabla^m w') - \bar{M}(t)\left(\nabla^m (u+v), \nabla^m w'\right)\left(\nabla^m (u+v), \nabla^m w\right) \\ & -\varepsilon\left(M(t)\left\|\nabla^m w\right\|^2 + \bar{M}(t)(\nabla^m (u+v), \nabla^m w)^2\right) + \varepsilon\left\|w'\right\|^2. \end{aligned} \tag{3.37}$$

Obviously, there exists $b \ge a > 0$ and $C_4 > 0$, such that

$$a\left(\left\|w'\right\|^2 + \left\|\nabla^m w\right\|^2\right) \le H(t) \le b\left(\left\|w'\right\|^2 + \left\|\nabla^m w\right\|^2\right), \tag{3.38}$$

$$K(t) \le \frac{1}{8}\left\|\nabla^m w'\right\|^2 + C_4\left\|\nabla^m w\right\|^2. \tag{3.39}$$

By (H_4), we know $H_0^m(\Omega) \subset L^{p+1}(\Omega)$. So we have

$$\begin{aligned} \left|-\left(g(u) - g(v), w' + \varepsilon w\right)\right| &\le C_1 \int_\Omega \left(|u|^{p-1} + |v|^{p-1}\right)|w|\left(|w'| + \varepsilon|w|\right)dx \\ &\le C_1\left(\|u\|_{p+1}^{p-1} + \|v\|_{p+1}^{p-1}\right)\|w\|_{p+1}\left(\|w'\|_{p+1} + \varepsilon\|w\|_{p+1}\right) \\ &\le \frac{1}{8}\left\|\nabla^m w'\right\|^2 + C_5\left(\left\|w'\right\|^2 + \left\|\nabla^m w\right\|^2\right). \end{aligned} \tag{3.40}$$

By (G_3), we get

$$\begin{aligned} \int_\tau^t \left\|h(s,u_s) - h(s,v_s)\right\|^2 ds &\le C_h^2 \int_{\tau-r}^t \left\|u(s) - v(s)\right\|^2 ds \\ &\le \lambda_1^{-m} C_h^2 r \left\|\phi - \psi\right\|_{C_{V,H}}^2 + \lambda_1^{-m} C_h^2 \int_\tau^t \left\|\nabla^m w(s)\right\|^2 ds, \end{aligned} \tag{3.41}$$

$$\begin{aligned} \left(h(t,u_t) - h(t,v_t), w' + \varepsilon w\right) &\le \left\|h(t,u_t) - h(t,v_t)\right\|^2 + \frac{1}{2}\left\|w'\right\|^2 + \frac{\varepsilon}{2}\left\|w\right\|^2 \\ &\le \left\|h(t,u_t) - h(t,v_t)\right\|^2 + \frac{1}{2}\left\|w'\right\|^2 + \frac{\varepsilon}{2\lambda_1^m}\left\|\nabla^m w\right\|^2 \\ &\le \left\|h(t,u_t) - h(t,v_t)\right\|^2 + \frac{1}{2}\left(\left\|w'\right\|^2 + \left\|\nabla^m w\right\|^2\right), \end{aligned} \tag{3.42}$$

with $0 < \varepsilon < \lambda_1^m$.

Inserting (3.38)-(3.42) into (3.35), we obtain

$$\frac{d}{dt}H(t) + \frac{3}{4}\left\|\nabla^m w'\right\|^2 \le \left\|h(t,u_t) - h(t,v_t)\right\|^2 + \left(C_4 + C_5 + \frac{1}{2}\right)\left(\left\|w'\right\|^2 + \left\|\nabla^m w\right\|^2\right). \tag{3.43}$$

By (3.38), (3.41), integrating (3.43) over (τ, t), we can get

$$\begin{aligned} & a\left(\left\|w'\right\|^2 + \left\|\nabla^m w\right\|^2\right) + \frac{3}{4}\int_\tau^t \left\|\nabla^m u(s)\right\|^2 ds \\ & \le b\left(\left\|w'(\tau)\right\|^2 + \left\|\nabla^m w(\tau)\right\|^2\right) + \left(C_4 + C_5 + \frac{1}{2}\right)\int_\tau^t \left(\left\|w'(s)\right\|^2 + \left\|\nabla^m w(s)\right\|^2\right)ds + \lambda_1^{-m} C_h^2 r \left\|\phi - \psi\right\|_{C_{V,H}}^2 + \lambda_1^{-m} C_h^2 \int_\tau^t \left\|\nabla^m w(s)\right\|^2 ds \\ & \le \left(b + \lambda_1^{-m} C_h^2 r\right)\left\|\phi - \psi\right\|_{C_{V,H}}^2 + \left(C_4 + C_5 + \lambda_1^{-m} C_h^2 + \frac{1}{2}\right)\int_\tau^t \left(\left\|w'(s)\right\|^2 + \left\|\nabla^m w(s)\right\|^2\right)ds. \end{aligned} \tag{3.44}$$

Combining the Gronwall lemma, we get

$$\left\|w'\right\|^2 + \left\|\nabla^m w\right\|^2 \le \frac{b + \lambda_1^{-m} C_h^2 r}{a}\left\|\phi - \psi\right\|_{C_{V,H}}^2 e^{\frac{2C_4 + 2C_5 + 1}{2a}(t-\tau)}. \tag{3.45}$$

If ϕ and ψ stand for the same initial value, there has

$$\left\|w'\right\|^2 + \left\|\nabla^m w\right\|^2 \le 0. \tag{3.46}$$

Therefore, $u = v$.

Step 3: Next, we need the further estimate of $\int_t^\infty \|\nabla^m u_t(s)\|^2 ds$.

Multiplying (1.1) by $2u'$ gives

$$\frac{d}{dt}\left[\|u'\|^2 + \frac{1}{(q+1)\beta}\left(\alpha + \beta\|\nabla^m u\|^2\right)^{q+1} + 2J(u) - 2(f(x),u)\right] + 2\|\nabla^m u'\|^2 = 0. \tag{3.47}$$

Integrating the above equality over $(t, \inf)$. So, there exists $C_3 > 0$, such that

$$\int_t^\infty \|\nabla^m u_t(s)\|^2 ds \le C_3, t \ge \tau. \tag{3.48}$$

IV. Existence of the Pullback Attractor

In this subsection, we assume that $f \in H$, we aim to study the pullback attractor for the initial value problem (1.1).

From Theorem 3.1, the initial value problem (1.1) generates a family two-parameter semigroup $U(\cdot,\cdot)$ in $C_{V,H}$, which can be defined by

$$U(t,\tau)(\phi) = u_t(\cdot;\tau,\phi), t \ge \tau, \phi \in C_{V,H}. \tag{4.1}$$

Lemma 4.1. Let $\phi, \psi \in C_{V,H}$ be the two initial values for the problem (1.1), and $\tau \in R$ is the initial time. Denote by $u(\cdot) = u(\cdot;\tau,\phi)$ and $v(\cdot) = v(\cdot;\tau,\psi)$ the corresponding solution to (1.1). Then, there exists a constant $C_6 > 0$ which does not depend on the initial data and time, such that

$$\|\nabla^m u(t) - \nabla^m v(t)\|^2 + \|u'(t) - v'(t)\|^2 \le \frac{b + \lambda_1^{-m} C_h^2 r}{a}\|\phi - \psi\|_{C_{V,H}}^2 e^{C_6(t-\tau)}, \forall t \ge \tau, \tag{4.2}$$

and

$$\|u_t - v_t\|_{C_{V,H}}^2 \le \frac{b + \lambda_1^{-m} C_h^2 r}{a}\|\phi - \psi\|_{C_{V,H}}^2 e^{C_6(t-\tau)}, \forall t \ge \tau + r, \tag{4.3}$$

with $a, b > 0$ are given in (3.38).

Proof. We denote $w = u - v$ .By(3.32),we can get (4.2) easily with $C_6 = \dfrac{2C_4 + 2C_5 + 1}{2a}$ in (3.46). If we consider $t \ge \tau + r$, then $t + \theta \ge \tau$ for any $\theta \in [-r, 0]$, and

$$\begin{aligned}\|\nabla^m w(t+\theta)\|^2 + \|w'(t+\theta)\|^2 &\le \frac{b + \lambda_1^{-m} C_h^2 r}{a}\|\phi - \psi\|_{C_{V,H}}^2 e^{C_6(t-\tau+\theta)} \\ &\le \frac{b + \lambda_1^{-m} C_h^2 r}{a}\|\phi - \psi\|_{C_{V,H}}^2 e^{C_6(t-\tau)}, \forall t \ge \tau + r.\end{aligned} \tag{4.4}$$

Thus,

$$\|w_t\|^2 \le \frac{b + \lambda_1^{-m} C_h^2 r}{a}\|\phi - \psi\|_{C_{V,H}}^2 e^{C_6(t-\tau)}, \forall t \ge \tau + r. \tag{4.5}$$

Theorem 4.1. The mapping $U(t,\tau): C_{V,H} \to C_{V,H}$ is continuous for any $t \ge \tau$.

Proof. Let $\phi, \psi \in C_{V,H}$ be the initial value for the problem (1.1) and $t \ge \tau$. Denote by $u(\cdot) = u(\cdot;\tau,\phi)$ and $v(\cdot) = v(\cdot;\tau,\psi)$ the corresponding solution to (1.1). Then, writing again $w = u - v$, we obtain the following:

If $t \in [\tau - r, \tau]$, then $w(t) = \phi(t-\tau) - \psi(t-\tau)$ and

$$\begin{aligned}\|\nabla^m w(t)\|^2 + \|w'(t)\|^2 &\le \|\phi - \psi\|_{C_V}^2 + \|\phi' - \psi'\|_{C_H}^2 \\ &\le \frac{b + \lambda_1^{-m} C_h^2 r}{a}\|\phi - \psi\|_{C_{V,H}}^2 e^{C_6(t-\tau+r)}.\end{aligned} \tag{4.6}$$

Thus, we have

$$\|\nabla^m w(t)\|^2 + \|w'(t)\|^2 \le \frac{b + \lambda_1^{-m} C_h^2 r}{a}\|\phi - \psi\|_{C_{V,H}}^2 e^{C_6(t-\tau+r)}, \forall t \ge \tau - r, \tag{4.7}$$

whence

$$\|w_t\|^2 \le \frac{b + \lambda_1^{-m} C_h^2 r}{a}\|\phi - \psi\|_{C_{V,H}}^2 e^{C_6(t-\tau+r)}, \forall t \ge \tau, \tag{4.8}$$

which implies the continuity of $U(t,\tau)$.

Theorem 4.2. Assume that $f \in H, \phi \in C_{V,H}$ and $g(u)$ satisfies $(H_1)-(H_3)$, h satisfies $(G_1)-(G_5)$ with $k_0 > 0$, and

$$\left(\frac{2C_h^{\ 2}}{C_2}\right)^{\frac{1}{4}} < \min\left\{2, \frac{q\lambda_1^{\ m}}{2}, \frac{\beta^q \lambda_1^{\ m}}{2}, \frac{-2-C_2-\lambda_1^{\ m}+\sqrt{(2+C_2+\lambda_1^{\ m})^2+16\lambda_1^{\ m}}}{4}\right\}. \tag{4.9}$$

Then, there exists a family $\{B(t)\}_{t\in R}$ of bounded sets in $C_{V,H}$ which is uniformly pullback absorbing for the process $U(\cdot,\cdot)$. Moreover, $B(t) = B^0$ for all $t \in R$, where B^0 is a bounded set in $C_{V,H}$.

Proof. By lemma 3.1, we know (3.30)-(3.31) for $t \ge \tau$ and $t \ge \tau - r$.

If we take now $t \ge \tau + r$, then for all $\theta \in [-r, 0]$ we have $t+\theta \ge \tau$ and so

$$\|\nabla^m u(t+\theta;\tau,\phi)\|^2 + \|u'(t+\theta;\tau,\phi)\|^2 \le \frac{\rho_0^{\ 2}}{2} + \hat{\rho}_0^{\ 2} d^2 e^{kr} e^{k(\tau-t)}, \tag{4.10}$$

or, in other words,

$$\|U(t,\tau)\phi\|_{C_{V,H}}^2 \le \frac{\rho_0^{\ 2}}{2} + \hat{\rho}_0^{\ 2} d^2 e^{kr} e^{k(\tau-t)}, \forall t \ge \tau + r, \phi \in D. \tag{4.11}$$

Therefore, there exists $T_D \ge r$ such that

$$\|U(t,t-s)\phi\|_{C_{V,H}}^2 \le \rho_0^{\ 2}, \forall t \in R, s \ge T_D, \phi \in D, \tag{4.12}$$

which means that the ball $B_{C_{V,H}}(0,\rho_0) = B^0 \subset C_{V,H}$ is uniformly pullback absorbing for the process $U(\cdot,\cdot)$.

Remark (See [2]) On the one hand, observe that if $t_0 \in R$ and $t \ge t_0$, then $u\left(t+\theta;t_0-s,\phi\right) = u\left(t+\theta;t-(s+t-t_0),\phi\right)$ and $u'\left(t+\theta;t_0-s,\phi\right) = u'\left(t+\theta;t-(s+t-t_0),\phi\right)$ with $s+t-t_0 \ge s$. As a consequence of (4.12), we have

$$\left\|U\left(t,t_0-s\right)\phi\right\|_{C_{V,H}}^2 \le \rho_0^2, \forall t_0 \in R, t \ge t_0, s \in T_D, \phi \in D, \tag{4.13}$$

or equivalently, we have $\forall t_0 \in R, t \ge t_0, \theta \in [-r,0], s \in T_D, \phi \in D,$

$$\left\|\nabla^m u\left(t+\theta;t_0-s,\phi\right)\right\|^2 + \left\|u'\left(t+\theta;t_0-s,\phi\right)\right\|^2 \le \rho_0^2. \tag{4.14}$$

On the other hand, (3.30) implies, $\forall t_0 \in R, t \ge t_0, s \in R, t \in t_0 - s - r, \phi \in D,$

$$\left\|\nabla^m u\left(t+\theta;t_0-s,\phi\right)\right\|^2 + \left\|u'\left(t+\theta;t_0-s,\phi\right)\right\|^2 \le \rho_0^2 + \rho_0^2 d^2. \tag{4.15}$$

Theorem 4.3. In addition to the assumptions in Theorem 4.1. Then, there exists a compact set $B_2 \subset C_{V,H}$ which is uniformly pullback attracting for the process $U(\cdot,\cdot)$, and consequently, there exists the pullback attractor $\mathrm{A}(t)_{t\in R}$. Moreover, $\mathrm{A}(t)_{t\in R} \subset C_{H^{2m}\cap V,V}$ for all $t \in R$.

Proof. For each $\varepsilon \in R$, the norm $\|\phi\|_\varepsilon^2 = \|\phi\|_{C_V}^2 + \|\phi' + \varepsilon\phi\|_{C_H}^2, \phi \in C_{V,H}$ is equivalent to $\|\cdot\|_0 := \|\cdot\|_{C_{V,H}}$. This allows us to obtain absorbing ball for the original norm by proving the existence of absorbing balls for this new norm for some suitable value of ε. Indeed, let us denote $B_\varepsilon(0,\rho) = \left\{\phi \in C_{V,H} : \|\phi\|_\varepsilon < \rho\right\}$.

Noticing that for $C_7 = \max\left\{2, 1+2\varepsilon^2\lambda_1^{-m}\right\}$, it follows that

$$\|\phi\|_{C_{V,H}}^2 = \|\phi\|_{C_V}^2 + \|\phi' + \varepsilon\phi - \varepsilon\phi\|_{C_H}^2 \le \|\phi\|_{C_V}^2 + 2\|\phi' + \varepsilon\phi\|_{C_H}^2 + 2\varepsilon^2\|\phi\|_{C_H}^2 \le C_7\|\phi\|_\varepsilon^2, \tag{4.16}$$

then we have $B_\varepsilon(0,\rho) \subset B_0\left(0, C_7^{\frac{1}{2}}\rho\right)$.

Let $D \subset C_{V,H}$ be a bounded set, i.e. there exists $d > 0$ such that for any $\phi \in D$ it holds $\|\phi\|_\varepsilon^2 = \|\phi\|_{C_V}^2 + \|\phi' + \varepsilon\phi\|_{C_H}^2 \le d^2$, and so, $\|\phi\|_{C_{V,H}}^2 \le C_7 d^2$.

Denote, as usual, by $u(\cdot) = u(\cdot;\tau,\phi)$ the solution of problem (2.1), and consider the following problems:

$$v'' + (-\Delta)^m v' + \left(\alpha + \beta\left\|\nabla^m u\right\|^2\right)^q (-\Delta)^m v + g(u) = f(x) + h(t,u_t), t \ge \tau,$$
$$v(t) = 0, t \in [\tau - r, \tau], \quad (4.17)$$
$$v'(t) = 0, t \in [\tau - r, \tau],$$

$$w'' + (-\Delta)^m w' + \left(\alpha + \beta\left\|\nabla^m u\right\|^2\right)^q (-\Delta)^m w = 0, t \ge \tau,$$
$$w(t) = \phi(t-\tau), t \in [\tau - r, \tau], \quad (4.18)$$
$$w'(t) = \phi'(t-\tau), t \in [\tau - r, \tau].$$

From the uniqueness of the solution of problem (2.1), (4.17) and (4.18) it follows that

$$u(\cdot) = v(\cdot) + w(\cdot), \forall t \in R, \text{ and } \forall t \ge \tau - r. \quad (4.19)$$

Consequently, $U(t,\tau)$ can be written as

$$U(t,\tau)(\phi) = U_1(t,\tau)(\phi) + U_2(t,\tau)(\phi), \phi \in C_{V,H}, t \ge \tau - r. \quad (4.20)$$

where $U_1(t,\tau)(\phi) = v_t(\cdot) = v_t(\cdot;\tau,\phi)$ and $U_2(t,\tau)(\phi) = w_t(\cdot) = w_t(\cdot;\tau,\phi)$ are the solution of (4.17) and (4.18) respectively.

First,thanks to (3.30), but with $g = f = h = 0$. Then, there exists $C_8 = C_8(\rho_0, d, \alpha) > 0$, it follows

$$\left\|\nabla^m w(t;\tau,\phi)\right\|^2 + \left\|w'(t;\tau,\phi)\right\|^2 \le C_8 d^2, \forall t \ge \tau, \phi \in D, \quad (4.21)$$

and by means of (4.10), then

$$\left\|w_t(\cdot;\tau,\phi)\right\|_{C_V}^2 + \left\|w_t'(\cdot;\tau,\phi)\right\|_{C_H}^2 \le C_8 d^2 e^{kr} e^{k(\tau-t)}, \forall t \ge \tau + r, \phi \in D. \quad (4.22)$$

Furthermore, for $t_0 \in R, t \ge t_0$ and $s > T_D \ge r$,

$$w\left(t;t_0 - s,\phi\right) = w\left(t;t-(s+t-t_0),\phi\right), \quad (4.23)$$

with $s + t - t_0 > s \ge T_D \ge r$.

Thus, (4.22) implies in particular

$$\left\|w\left(t;t_0 - s,\phi\right)\right\|^2 \le C_8 d^2 e^{kr} e^{k(t_0 - s - t)} \le C_8 d^2 e^{kr} e^{-ks}, \forall t_0 \in R, t \ge t_0, s \ge T_D, \phi \in D. \quad (4.24)$$

Then, (4.22) yields that

$$\left\|U_2(t,t-s)\phi\right\|_{C_{V,H}}^2 \le C_8 d^2 e^{kr} e^{-ks}, \forall t \in R, s \ge r, \phi \in D. \quad (4.25)$$

Whence

$$\lim_{s \to +\infty} \sup_{t \in R} \sup_{\phi \in D} \left\|U_2(t,t-s)\phi\right\|_{C_{V,H}}^2 = 0. \quad (4.26)$$

Let us now proceed with the other term. Let us fix $t_0 \in R, s \ge T_D, \phi \in D$ and denote

$$u(t) = u(t;t_0 - s,\phi), v(t) = v(t;t_0 - s,\phi), t \ge t_0 - s - r, \quad (4.27)$$

and

$$F(t) = -g(u) + f(x) + h(t,u_t), t \ge t_0 - s. \quad (4.28)$$

By (G_3), then

$$\left\|F(t)\right\| \le \left\|g(u)\right\| + \left\|f(x)\right\| + L_h \left\|u_t\right\|_{C_H}. \quad (4.29)$$

From (H_4), Sobolev imbedding theory and (4.14), there exists $C_9 = C_9(d,\rho_0) > 0$, such that

$$\left\|F(t)\right\| \le C_9 + \left\|f(x)\right\| + L_h \lambda_1^{\frac{-m}{2}} \rho_0 = C_{10}, \forall t \ge t_0, \quad (4.30)$$

and from (4.15), then

$$\left\|F(t)\right\| \le C_9 + \left\|f(x)\right\| + L_h \lambda_1^{\frac{-m}{2}} \left(\rho_0^2 + \hat{\rho}_0^2 d^2\right)^{\frac{1}{2}} \le C_{10} + L_h \lambda_1^{\frac{-m}{2}} \rho_0 d, \forall t \ge t_0 - s. \quad (4.31)$$

Let $q = v' + \varepsilon v$ with $0 < \varepsilon < \min\left\{\frac{\alpha^q}{4}, \frac{\alpha^q \lambda_1^m}{2}, \frac{-3 + \sqrt{9 + 4\lambda_1^m}}{2}\right\}$, then multiplying (4.17) by $(-\Delta)^m q$ gives

$$\frac{d}{dt}\left[\left\|\nabla^m q\right\|^2+\left[\left(\alpha+\beta\left\|\nabla^m u\right\|^2\right)^q-\varepsilon\right]\left\|\Delta^m v\right\|^2\right]+2\left\|\Delta^m q\right\|^2-2\varepsilon\left\|\nabla^m q\right\|^2+2\varepsilon^2\left(\nabla^m v,\nabla^m q\right)-2\varepsilon^2\left\|\Delta^m v\right\|^2$$
$$-2\left\|\Delta^m v\right\|^2\left[q\beta\left(\alpha+\beta\left\|\nabla^m u\right\|^2\right)^{q-1}\left(\nabla^m u,\nabla^m u'\right)\right]+2\varepsilon\left(\alpha+\beta\left\|\nabla^m u\right\|^2\right)^q\left\|\Delta^m v\right\|^2=2\left(F(t),(-\Delta)^m q\right). \quad (4.32)$$

In (4.32), by Holder inequality and Young's inequality, then

$$2\left(F(t),(-\Delta)^m q\right)\le\left\|F(t)\right\|^2+\left\|\Delta^m q\right\|^2\le C_{10}^2+\left\|\Delta^m q\right\|^2, \quad (4.33)$$

$$2\left\|\Delta^m q\right\|^2-2\varepsilon\left\|\nabla^m q\right\|^2+2\varepsilon^2\left(\nabla^m v,\nabla^m q\right)-2\varepsilon^2\left\|\Delta^m v\right\|^2+\varepsilon\left(\alpha+\beta\left\|\nabla^m u\right\|^2\right)^q\left\|\Delta^m v\right\|^2$$
$$\ge\left\|\Delta^m q\right\|^2+\left(\lambda_1^m-2\varepsilon-\varepsilon^2\right)\left\|\nabla^m q\right\|^2+\left(\frac{\alpha^q\varepsilon}{2}-2\varepsilon^2\right)\left\|\Delta^m v\right\|^2+\left(\frac{\alpha^q\lambda_1^m\varepsilon}{2}-\varepsilon^2\right)\left\|\nabla^m v\right\|^2 \quad (4.34)$$
$$\ge\left\|\Delta^m q\right\|^2+\left(\lambda_1^m-2\varepsilon-\varepsilon^2\right)\left\|\nabla^m q\right\|^2$$
$$\ge 0.$$

Setting

$$z(t)=\left\|\nabla^m q\right\|^2+\left[\left(\alpha+\beta\left\|\nabla^m u\right\|^2\right)^q-\varepsilon\right]\left\|\Delta^m v\right\|^2\ge\left\|\nabla^m q\right\|^2+\left(\alpha^q-\varepsilon\right)\left\|\Delta^m v\right\|^2>0, \quad (4.35)$$

then substituting (4.33)-(4.34) into (4.43), we have

$$\frac{d}{dt}z(t)+\left(\lambda_1^m-2\varepsilon-\varepsilon^2\right)\left\|\nabla^m q\right\|^2+\varepsilon\left(\alpha+\beta\left\|\nabla^m u\right\|^2\right)^q\left\|\Delta^m v\right\|^2$$
$$\le C_{10}^2+2q\beta\left(\alpha+\beta\left\|\nabla^m u\right\|^2\right)^{q-1}\left\|\nabla^m u\right\|\left\|\nabla^m u'\right\|\left\|\Delta^m v\right\|^2. \quad (4.36)$$

Therefore, by (3.31) and (4.36) for $t\ge t_0-s$, there exists $C_{11}=C_{11}(d,q,\alpha,\beta,\rho_0)>0$, such that

$$\frac{d}{dt}z(t)+\varepsilon z(t)\le C_{10}^2+C_{11}\left\|\nabla^m u'\right\|z(t). \quad (4.37)$$

Noticing that $y(t_0-s)=0$, and for (4.37) in $[t_0-s,t_0]$, by lemma 2.4 and (3.3), we obtain

$$z(t_0)\le C_{12}=C_{12}(L_h,d,\hat{\rho}_0), \quad (4.38)$$

and

$$z(t)\le C_{11}e^{C_3}z(t_0)e^{-\varepsilon(t-t_0)}+C_{10}^2\int_{t_0}^{t}e^{-\varepsilon(t-y)}dy\le C_{13}e^{-\varepsilon t}+\frac{C_{10}^2}{\varepsilon}. \quad (4.39)$$

Then, there exists $T_D'\ge T_D$ such that, if $s\ge T_D'$,

$$z(t)\le C_{13}e^{-\varepsilon t}+\frac{C_{10}^2}{\varepsilon},t_0\in R,t\ge t_0. \quad (4.40)$$

Recall that $z(t)=z(t;t_0-s,\phi)$, if we fix $t\ge t_0$, take $s=T_D'$ and denote $\tilde{s}=t-t_0+T_D'$, we have, provided t is large enough, that

$$z(t;t_0-T_D',\phi)=z(t;t-(t-t_0+T_D'),\phi)=z(t;t-\tilde{s},\phi)\le\frac{2C_{10}^2}{\varepsilon}. \quad (4.41)$$

In conclusion, there exists $T_D''>0$ such that for all $t\in R$, and $s\ge T_D'+T_D''$,

$$z(t;t-s,\phi)\le\frac{2C_{10}^2}{\varepsilon},\forall\phi\in D. \quad (4.42)$$

Denoting $\hat{T}_D=T_D'+T_D''+r$, we have for all $\phi\in D,t\in R,s\ge\hat{T}_D$,

$$\left\|\Delta^m v(t,t-s,\phi)\right\|^2+\left\|\nabla^m v'(t,t-s,\phi)\right\|^2\le\frac{2C_{10}^2}{\varepsilon}, \quad (4.43)$$

and, by repeating once more the same argument previously used,

$$\left\|v_t(\cdot;t-s,\phi)\right\|_{H^{2m}\cap V,V}^2\le\frac{2C_{10}^2}{\varepsilon}, \quad (4.44)$$

for all $\phi \in D, t \in R, s \geq \hat{T}_D$.

This means that the ball $B^1 = B_{H^{2m}\cap V,V}\left(0, \frac{2C_{10}^2}{\varepsilon}\right)$ is a bounded set in $H^{2m} \cap V, V$ which, in addition, is uniformly pullback absorbing for the family of operators $U_1(\cdot,\cdot)$. As B_1 is a bounded set in $C_{V,H}$, then there exists $T_{B^1} \geq r$ such that

$$U_1(t,t-s)B^1 \subset B^1, \forall t \in R, s \geq T_{B^1}, \tag{4.45}$$

and, therefore, the bounded set $B^2 \subset C_{H^{2m}\cap V,V}$ given by

$$B^2 = \bigcup_{t\in R}\bigcup_{s\geq T_{B_1}} U_1(t,t-s)B^1 \subset B^1, \tag{4.46}$$

is uniformly pullback absorbing for $U_1(\cdot,\cdot)$ in $C_{V,H}$.

By Ascoli-Arzela theorem, we can prove that $\bar{B}^2$ is compact, so $\{B(t) \equiv \bar{B}^2\}$ is a family of compact subsets in $C_{V,H}$, which is also uniformly pullback attracting for $U(\cdot,\cdot)$, and the proof has been completed.

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