International Journal of Modern Communication Technologies & Research (IJMCTR)
ISSN: 2321-0850, Volume-4, Issue-11, November 2016

Exponential attractors and inertial manifolds for the higher-order nonlinear Kirchhoff-type equation

Ling Chen, Wei Wang, Guoguang Lin

***Abstract*— In this paper, we consider the higher-order Kirchhoff-type equation with nonlinear strongly dissipation:**

$$u_{tt}+(-\Delta)^m u_t+\phi\left(\left\|\nabla^m u\right\|^2\right)(-\Delta)^m u+g(u)=f(x)$$

in n dimensional space. The squeezing property of the nonlinear semi-group associated with this equation and the existence of exponential attractors are proved. The inertial manifolds are also estimated. The main result is that the nonlinear $g(u)$ and damping coefficient $\phi(s)$ meet conditions $0<p\le\frac{2n}{n-2m}$, $n\ge 3$, and $\varepsilon\le m_0\le\phi(s)\le m_1=\frac{2\mu_1-1}{4}$. In this case, exponential attractors and inertial manifolds are established.

***Index Terms*— Higher-order; Squeezing property; Exponential attractors; Inertial manifold.**

I. INTRODUCTION

In this paper, we research the existence of exponential attractors and inertial manifolds for the higher-order Kirchhoff-type equation with nonlinear strongly dissipation:

$$u_{tt}+(-\Delta)^m u_t+\phi\left(\left\|\nabla^m u\right\|^2\right)(-\Delta)^m u+g(u)=f(x), \tag{1.1}$$

$x\in\Omega$, t>0,m>1

$$u(x,t)=0,\quad \frac{\partial^i u}{\partial v^i}=0,\quad i=1,2,\ldots,m-1,\ x\in\partial\Omega,\ t>0 \tag{1.2}$$

$$u(x,0)=u_0(x),\qquad u_t(x,0)=u_1(x) \tag{1.3}$$

Where Ω is a bounded domain of R^n, with a smooth Dirichlet boundary $\partial\Omega$ and initial data, the damping coefficient is function of the L_2-norm of the gradient m power, $g(u)$ is a nonlinear forcing, $(-\Delta)^m u_t$ is a strongly dissipation.

As well as we known, the exponential attractors and inertial manifolds occupies a significant position in the study of the long-time behavior of infinite-dimensional dynamical system, because of exponential attractors possess a deeper and more practical property, it is a compact invariant and exponentially attractor all the orbits of the solution.

While an inertial manifold is a finite-dimension invariant Lipschitz surface in the phase space of the system attracting all trajectories at an exponential rate. It play an important role between infinite-dimensional dynamical system and finite-dimensional dynamical system.

In [1], Zheng Songmu and Albert Milani studied the exponential attractors and inertial manifolds for the following singular perturbations of Cahn-Hilliard equations in one dimensional of space

$$\varepsilon u_{tt}+u_t+\Delta(\Delta u-u^3+u-\delta u_t)=0, \tag{1.4}$$

where $\varepsilon\ge 0,\delta\ge 0$. Their results allow that the dynamical systems generated by these two problems admit exponential attractors and inertial manifolds in the phase space $H_0^1\times H^{-1}(0,\pi)$.

In [2], Jingzhu Wu and Guoguang Lin research the existence of inertial manifolds of two-dimensional strong damping Boussinesq equation while $\alpha>2$.

$$u_{tt}-\alpha\Delta u_t-\Delta u+u^{2k+1}=f(x,y),\quad (x,y)\in\Omega, \tag{1.5}$$

$$u(x,y,0)=u_0(x,y),(x,y)\in\Omega, \tag{1.6}$$

$$u(x,y,t)=u(x+\pi,y,t)=u(x,y+\pi,t)=0,(x,y)\in\Omega \tag{1.7}$$

Where $\Omega=(0,\pi)\times(0,\pi)\subset R\times R, t>0$.

Recently, in [3], Zhijian Yang, Zhiming Liu study the existence of exponential attractor for the Kirchhoff equations with strong nonlinear damping and supercritical nonlinearity

$$u_{tt}-\sigma(\|\nabla u\|^2)\Delta u_t-\phi(\|\nabla u\|^2)\Delta u+f(u)=h(x), \tag{1.8}$$

$$u|_{\partial\Omega}=0,\ u(x,0)=u_0(x),\ u_t(x,0)=u_1(x),\ x\in\Omega. \tag{1.9}$$

Where Ω is a bounded domain in $\mathbb{R}^N$ with the smooth boundary $\partial\Omega$, the nonlinear functions $\sigma(s),\phi(s)$ and $f(u)$ and the external force term h will be specified later.

They acquire an exponential attractor in natural energy space by using a new method based on the weak quasi-stability estimates (rather than the strong one as usual). Chueshov[4] first studies the existence and uniqueness of solution and global attractors for problem(1.8)-(1.9). His results show that the growth exponential p of the nonlinearity term $f(u)$ need to be satisfied: $p_1<p<p_2$, with $p_1=\frac{N+2}{(N-2)^+}$, $p_2=\frac{N+4}{(N-4)^+}$. However, when $p\le p_1$, he obtained an exponential attractor by virtue of the strong quasi-stability estimates.

Ke Li, Zhijian Yang[5] studied the exponential attractors for the strongly damped wave equation

Ling Chen, Department of Mathematics, Yunnan University, Kunming, Yunnan 650091, People's Republic of China
Wei Wang, Department of Mathematics, Yunnan University, Kunming, Yunnan 650091, People's Republic of China
Guoguang Lin, Department of Mathematics, Yunnan University, Kunming, Yunnan 650091, People's Republic of China

$$u_{tt} - \Delta u_t - \Delta u + \varphi(u) = f, \tag{1.10}$$

$$u(0) = u_0, u_t(0) = u_t, \tag{1.11}$$

$$u|_{\partial\Omega} = 0, \tag{1.12}$$

Where $f \in L^2(\Omega), \varphi \in C^2(\Omega), \varphi(0) = 0.$

(A_1) $\liminf_{|r|\to\infty} \varphi'(r) > -\lambda_1, r \in R, \lambda_1$ is the first eigenvalue of $-\Delta$ with Dirichlet boundary condition, and φ is of critical growth.

$(A_2) \varphi''(r) \le c(1+|r|^3), r \in R.$

Under suitable assumption $(A_1),(A_2)$. They obtain exponential attractor by the l-trajectories method, and they give an explicit upper bound of the fractal dimension of the exponential attractor.

The problem (1.10)-(1.12) was studied by Meihua Yang and Chunyou Sun in [6] when their assumption respectively are:

$(H_1) \varphi \in C^1(R), \varphi(0) = 0$;

(H_2) growth condition $|\varphi'(s)| \le c(1+|s|^p)$;

(H_3) $\liminf_{|s|\to\infty} \frac{\varphi(s)}{s} > -\lambda_1, 0 \le p \le 4.$

They result allow that for each $T > 0$ fixed, there is a bound (in $H^2 \times H^1$) set which attracts exponentially every $H^1 \times L^2$ bounded set w.r.t the stronger $H^1 \times H^1$ -norm for all $t \ge T$ and has finite fractal dimension in $H_0^1 \times H_0^1$ for the case $f \in L^2(\Omega)$

Guigui Xu, Libo Wang and Guoguang Lin [7] studied global attractor and inertial manifold for the strongly damped wave equation

$$u_{tt} - \alpha\Delta u + \beta\Delta^2 u - \gamma\Delta u_t + g(u) = f(x,t), \quad (x,t) \in \Omega \times R^+, \tag{1.13}$$

$$u(x,0) = u_0(x), u_t(x,0) = u_1(x), x \in \Omega, \tag{1.14}$$

$$u|_{\partial\Omega} = 0, \Delta u|\partial\Omega = 0, (x,t) \in \partial\Omega \times R^+. \tag{1.15}$$

They give assumption for the nonlinearity term g satisfies the following inequality:

(H_1) $\liminf_{|s|\to\infty} \frac{G(s)}{s^2} \ge 0, s \in R, G(s) = \int_0^s g(r)dr.$

(H_2) There is some positive constant C_1 such that

$\liminf_{|s|\to\infty} \frac{sg(s) - C_1 G(s)}{s^2} \ge 0, s \in R.$

Under suitable assumption, they prove the dynamical system admits the inertial manifold by using the Hadamard's graph transformation method.

More research on exponential attractors and inertial manifolds, we can refer to literature [8-12].

In the present paper, the existence of the exponential attractor is easily to be proved. Concerning the inertial manifold, there exist lots of difficulties in the process of existence of inertial manifolds, in order to overcome the difficulties, we take advantage of Hadamard's graph transformation method. As a graph of a Lipschitz continuous function defined over a finite-dimensional subspace of X. Following Robinson [11], we first transform equation (1.1) into an equivalent first-order system of the form.

$$U_t + AU = F(U), \quad U \in X \tag{1.16}$$

So as to conquer the difficulties, we need some reasonable assumption in literature [12], now, we will need the assumption of display as follows:

(1) $g(u) \le C(1+|u|^p),\ 0 < p \le \frac{2n}{n-2m}, n \ge 3;$

(2) $\phi(s) \in C^1(R);$

(3) $\varepsilon \le m_0 \le \phi(s) \le m_1 = \frac{2\mu_1 - 1}{4}$

The other two new hypothesis is given:

(4) $(\phi - \gamma)(\nabla^m u, \nabla^m v) \ge (\nabla^m u, \nabla^m v),\ \gamma \in (\varepsilon, m_0);$

The main contributions of this paper are: (a) the problem considered in this paper is higher-order nonlinear equation with strongly damping term and this problem is representative; (b) the estimates are precise and the proofs are understood easily.

This paper is arranged as follows. In section 2, some notations and basic concepts are established. In section 3, it is proved that the exponential attractor exists. In section 4, we will discuss the existence of inertial manifolds.

II. PRELIMINARIES

For convenience, we first introduce the following notations:

$H = L^2(\Omega), V_1 = H_0^m(\Omega) \times L^2(\Omega),$

$V_2 = (H^{2m}(\Omega) \cap H_0^1(\Omega)) \times H_0^m(\Omega),$

$c_i (i = 0,1,2\ldots)$ denote various positive constants and they may be different at each appearance. where $(.\,,\,.)$ and $\|.\|$ are the inner product and norm of H. The inner product and the norm in V_1 space is defined as follows:

$\forall U_i \in (u_i, v_i) \in V_i, i = 1,2,$ we have

$$(U_1, U_2) = (\nabla^m u_1, \nabla^m u_2) + (v_1, v_2) \tag{2.1}$$

$$\|U\|_{V_1}^2 = (U,U)_{V_1} = \|\nabla^m u\|^2 + \|v\|^2 \tag{2.2}$$

Let $U = (u,v) \in V_1, v = u_t + \varepsilon u,$

$\frac{\gamma + \lambda_1^{-m}}{3} \le \varepsilon \le \min\left\{ \frac{\gamma}{\lambda_1^{\frac{m}{2}} + 2}, \sqrt[4]{2}\lambda_1^{\frac{m}{4}} \right\}$ Equation (1.1) is equivalent to the following the evolution equation

$$U_t + AU = F(U), \tag{2.3}$$

with

$$U = (u,v) \in V_1, v = u_t + \varepsilon u, \tag{2.4}$$

$$AU = \begin{pmatrix} \varepsilon u - v \\ \varepsilon^2 u - \varepsilon v + (-\Delta)^m v + (\phi(\|\nabla^m u\|^2) - \varepsilon)(-\Delta)^m u \end{pmatrix}, \tag{2.5}$$

$$F(U) = \begin{pmatrix} 0 \\ f(x) - g(u) \end{pmatrix} \tag{2.6}$$

We will use the following notations, let V_1, V_2 are two Hilbert spaces, we have $V_1 \mapsto V_2$ with dense and continuous injection, and $V_1 \mapsto V_2$ is compact. Let $S(t)$ is a map from V_i into V_i, $i=1,2$.

Definition 2.1.[15] The semi-group $S(t)$ possesses a (V_2,V_1)-compact attractor A, If it exists a compact set $A \subset V_1$, A attracts all bounded subsets of V_2, and $S(t)A = A, \forall t \geq 0$.

Definition 2.2.[16] A compact set M is called a (V_2,V_1)-exponential attractor for the system $(S(t),B)$, if $A \subseteq M \subseteq B$ and

1) $S(t)M \subseteq M$, $\forall t \geq 0$,

2) M has finite fractal dimension, $d_f(M) < +\infty$;

3)there exist positive constants c_2, c_3 such that $dist(S(t)B,M) \leq c_2 e^{-c_3 t}, t>0$,

where $dist_{V_1}(A,B) = \sup_{x\in A}\inf_{y\in B}|x-y|_{V_1}, B \subset V_1$ is a positive invariant set of $S(t)$.

Definition 2.3.[16] $S(t)$ is said to satisfy the discrete squeezing property on B if there exists an orthogonal projection P_N of rank N such that for every u and v in B, either

$$|S(t_*)u - S(t_*)v|_{V_1} \leq \delta |u-v|_{V_1}, \delta \in (0,\tfrac{1}{8}),$$

or

$$|Q_N(S(t_*)u - S(t_*)v)|_{V_1} \leq |P_N(S(t_*)u - S(t_*)v)|_{V_1},$$

where $Q_N = I - P_N$.

Theorem 2.1. Assume that

1) $S(t)$ possesses a (V_2,V_1)-compact attractor A;

2) it exists a positive invariant compact set $B \subset V_1$ of $S(t)$;

3) $S(t)$ is a Lipschitz continuous map with Lipschitz constant l on B, and satisfies the discrete squeezing property on B.

Then $S(t)$ has a (V_2,V_1)-exponential attractor $M \supseteq A$ on B, and $M = \bigcup_{0\leq t\leq t_*} S(t)M_*$,

$M_* = A \bigcup (\bigcup_{j=1}^{\infty}\bigcup_{k=1}^{\infty} S(t_*)^j (E^{(k)}))$. Moreover, the fractal dimension of M satisfies $d_f(M) \leq cN_0 + 1$, where $N_0, E^{(\kappa)}$ are defined as in [17].

Proposition 2.1.[16] There is $t_0(B_0)$ such that $B = \overline{\bigcup_{0\leq t\leq t_0} S(t)B_0}$ is the positive invariant set of $S(t)$ in V_1, and B attracts all bounded subsets of V_2, where B_0 is a closed bounded absorbing set for $S(t)$ in V_2.

Proposition 2.2. Let B_0, B_1 respectively are closed bounded absorbing set of (2.3) in V_2, V_1, then $S(t)$ possesses a (V_2,V_1)-compact attractor A.

Definition 2.4.[18] An inertial manifold μ is a finite-dimensional manifold enjoying the following three properties:

1) μ is Lipschitz,

2) μ is positively invariant for the semi-group $\{S(t)\}_{t\geq 0}$, i.e. $S(t)\mu \subset \mu$, $\forall t \geq 0$,

3) μ attracts exponentially all the orbits of the solution.

Definition 2.5.[13] Let $A: X \to X$ be an operator and assume that $F \in C_b(X,X)$ satisfies the Lipschitz condition $\|F(U)-F(V)\|_X \leq l_F \|U-V\|_X$, $U,V \in X$.

The operator A is said to satisfy the spectral gap condition relative to F, if the point spectrum of the operator A can be divided into two parts σ_1 and σ_2, of which σ_1 is finite, and such that, if

$$\Lambda_1 = \sup\{\mathrm{Re}\,\lambda \,|\, \lambda \in \sigma_1\},\ \Lambda_2 = \inf\{\mathrm{Re}\,\lambda \,|\, \lambda \in \sigma_2\}, \tag{2.7}$$

and

$$X_i = span\{\omega_j \,|\, j \in \sigma_i\}, i=1,2. \tag{2.8}$$

Then

$$\Lambda_2 - \Lambda_1 > 4l_F \tag{2.9}$$

and the orthogonal decomposition

$$X = X_1 \oplus X_2 \tag{2.10}$$

holds with continuous orthogonal projections $P_1: X \to X_1$ and $P_2: X \to X_2$.

Lemma 2.1.[1] Let the eigenvalues $\mu_j^{\pm}$, $j \geq 1$ be arranged in nondecreasing order, for all $m \in N$, there is $N \geq m$ such that μ_N^- and μ_{N+1}^- are consecutive.

III. Exponential Attractors

Theorem 3.1.[14] Under of the proper assume for $g(u), \phi(u)$, the initial boundary value problem (1.1)-(1.3) with Dirichlet boundary exists unique smooth solution. This solution possesses the following properties:

$$\|(u,v)\|_{V_1}^2 = \|\nabla^m u\|^2 + \|v\|^2 \leq c(R_0),$$

$$\|(u,v)\|_{V_2}^2 = \|\Delta^m u\|^2 + \|\nabla^m v\|^2 \leq c(R_1).$$

We denote the solution in Theorem 2.1, by $S(t)(u_0,v_0) = (u(t),v(t))$, the $S(t)$ is a continuous semi-group in V_1, we have the ball:

$$B_1 = \{(u,v) \in V_1 : \|(u,v)\|_{V_1}^2 \leq c(R_0)\},$$

$$B_0 = \{(u,v) \in V_2 : \|(u,v)\|_{V_2}^2 \leq c(R_1)\}.$$

Respectively is a absorbing set of $S(t)$ in V_1 and V_2.

Lemma 3.1. For any $U = (u,v) \in V_1$, we have

$$(AU,U) \geq k_1\|U\|_{V_1}^2 + k_2\|\nabla^m v\|^2.$$

Proof. By (2.1) and (2.2), we obtain

$$(AU,U)=(\nabla^m(\varepsilon u-v),\nabla^m u)+(A,v)$$
$$=\varepsilon\left\|\nabla^m u\right\|^2-(\nabla^m v,\nabla^m u)+(A,v) \tag{3.1}$$

Where $A=-\varepsilon v+\varepsilon^2 u+(\phi(\left\|\nabla^m u\right\|^2)-\varepsilon)(-\Delta)^m u+(-\Delta)^m v$

$$(A,v)=(-\varepsilon v+\varepsilon^2 u+(\phi(\left\|\nabla^m u\right\|^2)-\varepsilon)(-\Delta)^m u+(-\Delta)^m v,v)$$
$$=-\varepsilon\left\|v\right\|^2+\varepsilon^2(u,v)+(\phi(\left\|\nabla^m u\right\|^2)-\varepsilon)(\nabla^m u,\nabla^m v)+\left\|\nabla^m v\right\|^2 \tag{3.2}$$

By using a new hypothesis (4). Holder inequality, Young's inequality and Poincare inequality, we can work out the following terms

$$(\phi(\left\|\nabla^m u\right\|^2)-\varepsilon)(\nabla^m u,\nabla^m v)$$
$$=(\phi-\gamma)(\nabla^m u,\nabla^m v)+(\gamma-\varepsilon)(\nabla^m u,\nabla^m v) \tag{3.3}$$
$$\ge(\nabla^m u,\nabla^m v)+\frac{\gamma-\varepsilon}{2}\lambda_1^{\frac{m}{2}}\left\|v\right\|^2-\frac{\gamma-\varepsilon}{2}\left\|\nabla^m u\right\|^2$$

$$\varepsilon^2(u,v)\ge-\varepsilon^2\lambda_1^{-m}\left\|\nabla^m u\right\|\left\|\nabla^m v\right\|$$
$$\ge-\varepsilon^2\lambda_1^{-m}(\frac{1}{2\varepsilon^2}\left\|\nabla^m u\right\|^2+\frac{\varepsilon^2}{2}\left\|\nabla^m v\right\|^2) \tag{3.4}$$
$$=-\frac{1}{2}\lambda_1^{-m}\left\|\nabla^m u\right\|^2-\frac{1}{2}\lambda_1^{-m}\varepsilon^4\left\|\nabla^m v\right\|^2$$

Where $\lambda_1(>0)$ is the first eigenvalue of the operator $-\Delta$.

To sum up (3.2)-(3.4), we can receive

$$(AU,U)\ge(\frac{\gamma-\varepsilon}{2}\lambda_1^{\frac{m}{2}}-\varepsilon)\left\|v\right\|^2+(\frac{3\varepsilon-\gamma}{2}-\frac{\lambda_1^{-m}}{2})\left\|\nabla^m u\right\|^2+(1-\frac{\lambda_1^{-m}\varepsilon^4}{2})\left\|\nabla^m v\right\|^2$$

because of

$$\frac{\gamma+\lambda_1^{-m}}{3}\le\varepsilon\le\min\left\{\frac{\gamma}{\lambda_1^{\frac{m}{2}}+2},\sqrt[4]{2}\lambda_1^{\frac{m}{4}}\right\}$$

then

$$(\frac{\gamma-\varepsilon}{2}\lambda_1^{\frac{m}{2}}-\varepsilon)\ge0,(\frac{3\varepsilon-\gamma}{2}-\frac{\lambda_1^{-m}}{2})\ge0,(1-\frac{\lambda_1^{-m}\varepsilon^4}{2})\ge0$$

so, we can obtain

$$(AU,U)\ge k_1\left\|U\right\|_{V_1}^2+k_2\left\|\nabla^m v\right\|^2,$$

where

$$k_1=\min\{(\frac{\gamma-\varepsilon}{2}\lambda_1^{\frac{m}{2}}-\varepsilon),(\frac{3\varepsilon-\gamma}{2}-\frac{\lambda_1^{-m}}{2})\},k_2=(1-\frac{\lambda_1^{-m}\varepsilon^4}{2})\ge0$$

The proved is ended.

Let

$S(t)U_0=U(t)=(u(t),v(t))^T$, where $v=u_t(t)+\varepsilon u(t)$;

$S(t)V_0=V(t)=(\tilde{u}(t),\tilde{v}(t))^T$, where $\tilde{v}=\tilde{u}_t(t)+\varepsilon\tilde{u}(t)$;

Set

$W(t)=S(t)U_0-S(t)V_0=U(t)-V(t)=(w(t),z(t))^T$,

where $z(t)=w_t(t)+\varepsilon w(t)$, then $W(t)$ satisfies:

$$W_t(t)+AU-AV+(0,g(u)-g(\tilde{u}))^T=0, \tag{3.5}$$

$$W(0)=U_0-V_0. \tag{3.6}$$

For the sake of attesting (1.1)-(1.3) has a exponential attractor, we first prove the dynamical system $S(t)$ of (1.1)-(1.3) is Lipschitz continuous on B.

Lemma 3.2. (Lipschitz property) For any $U_0,V_0\in B,T\ge0$, we have

$$\left\|S(t)U_0-S(t)V_0\right\|_{V_1}^2\le e^{kt}\left\|U_0-V_0\right\|_{V_1}^2.$$

Proof. Applying the inner product of the equation (3.7) with $W(t)$ in V_1, we discover that

$$\frac{1}{2}\frac{d}{dt}\left\|W(t)\right\|^2+(AU-AV,W(t))+(z(t),g(u)-g(\tilde{u}))=0$$

(3.7)

Similar to Lemma 3.1, we obtain

$$(AU-AV,W(t))_{V_1}\ge k_1\left\|W(t)\right\|_{V_1}^2+k_2\left\|\nabla^m z(t)\right\|_{V_1}^2. \tag{3.8}$$

Using Young's inequality, Poincare inequality and Lagrange mean value theorem, we have

$$\left|(g(u)-g(\tilde{u}),z(t))\right|\le\left|g'(\xi)\right|\left\|w(t)\right\|\left\|z(t)\right\|$$
$$\le c_4\lambda_1^{-\frac{m}{2}}\left\|\nabla^m w(t)\right\|\left\|z(t)\right\|$$
$$\le\frac{c_4\lambda_1^{-\frac{m}{2}}}{2}(\left\|\nabla^m w(t)\right\|^2+\left\|z(t)\right\|^2) \tag{3.9}$$
$$=\frac{c_4\lambda_1^{-\frac{m}{2}}}{2}\left\|W(t)\right\|^2.$$

So we have

$$\frac{d}{dt}\left\|W(t)\right\|^2+2k_1\left\|W(t)\right\|^2+2\left\|\nabla^m z(t)\right\|^2$$
$$\le c_4\lambda_1^{\frac{-m}{2}}\left\|W(t)\right\|^2. \tag{3.10}$$

By using Gronwall inequality, we obtain

$$\left\|W(t)\right\|^2\le e^{c_4\lambda_1^{\frac{-m}{2}}}\left\|W(0)\right\|^2=e^{kt}\left\|W(0)\right\|^2, \tag{3.11}$$

where $k=c_4\lambda_1^{\frac{-m}{2}}$, so we have

$$\left\|S(t)U_0-S(t)V_0\right\|_{V_1}^2\le e^{kt}\left\|U_0-V_0\right\|_{V_1}^2.$$

The proved is ended.

Now, we define the operator $A=-\Delta:D(A)\to H$;

$D(A)=\{u\in H\left|A^m u\in H\right.\}$.

Obviously, A is an unbounded self-adjoint positive operator and A^{-1} is compact, we find by elementary spectral theory the existence of an orthogonal basis of H consisting of eigenvectors

ω_j of A such that

$$A\omega_j=\lambda_j\omega_j \quad 0<\lambda_1\le\lambda_2\le\cdots\le\lambda_j\to+\infty.$$

$\forall N$ denote by $P=P_n:\ \mathrm{H}\to\mathrm{span}\{\omega_1,\cdots\omega_N\}$ the projector,

$Q=Q_N=I-P_N$.

Next, we will use

$$\left\|A^m u\right\|=\left\|(-\Delta)^m u\right\|\ge\lambda_{n+1}\left\|u\right\|,\forall u\in Q_n(H^{2m}(\Omega)\cap H_0^1(\Omega)),$$

$$\left\|Q_n u\right\|\le\left\|u\right\|,u\in H$$

Lemma 3.3. For any $U_0, V_0 \in B$, let $Q_{n_0}(t) = Q_{n_0}(U(t) - V(t)) = Q_{n_0}W(t) = (w_{n_0}, z_{n_0})^T$, then

$$\left\|W_{n_0}(t)\right\|_{V_1}^2 \le \left(e^{-2k_1 t} + \frac{c_4 \lambda_{n_0+1}^{-\frac{m}{2}}}{2k_1 + k} e^{kt}\right)\left\|W(0)\right\|^2.$$

Detailed proof of Lemma 3.3, please refer to Lemma 3.3 of literature [11]; so we will omit it.

Lemma 3.4. (Discrete squeezing property) For any $U_0, V_0 \in B$, if

$$\left\|P_{n_0}(S(T^*)U_0 - S(T^*)V_0)\right\|_{V_1} \le \left\|(I - P_{n_0})(S(T^*)U_0 - S(T^*)V_0)\right\|_{V_1}$$

then we have

$$\left\|S(T^*)U_0 - S(T^*)V_0\right\|_{V_1} \le \frac{1}{8}\left\|U_0 - V_0\right\|_{V_1}.$$

Proof. If

$$\left\|P_{n_0}(S(T^*)U_0 - S(T^*)V_0)\right\|_{V_1} \le \left\|(I - P_{n_0})(S(T^*)U_0 - S(T^*)V_0)\right\|_{V_1}$$

then

$$\begin{aligned}
&\left\|S(T^*)U_0 - S(T^*)V_0\right\|^2 \\
&\le \left\|(I - P_{n_0})(S(T^*)U_0 - S(T^*)V_0)\right\|_{V_1}^2 \\
&+ \left\|P_{n_0}(S(T^*)U_0 - S(T^*)V_0)\right\|_{V_1}^2 \qquad (3.12)\\
&\le 2\left\|(I - P_{n_0})(S(T^*)U_0 - S(T^*)V_0)\right\|_{V_1}^2 \\
&\le 2\left(e^{-2k_1 T^*} + \frac{c_4 \lambda_{n_0+1}^{-\frac{m}{2}}}{2k_1 + k} e^{kT^*}\right)\left\|U_0 - V_0\right\|^2.
\end{aligned}$$

Let T^* be large enough

$$e^{-2k_1 T^*} \le \frac{1}{256}. \qquad (3.13)$$

Also let n_0 be large enough

$$\frac{c_4 \lambda_{n_0+1}^{-\frac{m}{2}}}{2k_1 + k} e^{kT^*} \le \frac{1}{256}. \qquad (3.14)$$

Substituting (3.13),(3.14) into (3.12), we obtain

$$\left\|S(T^*)U_0 - S(T^*)V_0\right\|_{V_1} \le \frac{1}{8}\left\|U_0 - V_0\right\|_{V_1}. \qquad (3.15)$$

Lemma 3.4 is proved.

Theorem 3.2. Under of the above assume, $(u_0, v_0) \in V_k$, $k = 1, 2$, $f \in H, v = u_t + \varepsilon u, \frac{\gamma + \lambda_1^{-m}}{3} \le \varepsilon \le \min\left\{\frac{\gamma}{\lambda_1^{\frac{m}{2}} + 2}, \sqrt[4]{2}\lambda_1^{\frac{m}{4}}\right\}$ t hen the initial boundary value problem (1.1)-(1.3) the solution semi-group has a (V_2, V_1) -exponential attractor on B, $M = \bigcup_{0 \le t \le T^*} S(t)(A \cup (\bigcup_{j=1}^{\infty} \bigcup_{k=1}^{\infty} S(T^*)^j (E^{(k)})))$, and the fractal dimension is satisfied $d_f(M) \le 1 + cN_0$.

Proof. According to Theorem 2.1, Lemma 3.2, Lemma 3.4, Theorem 3.2 is easily proven.

IV. INERTIAL MANIFOLD

Equation (1.1) is equivalent to the following one order evolution equation

$$U_t + HU = F(U), \qquad (4.1)$$

where

$$U = (u, v), v = u_t, H = \begin{pmatrix} 0 & -I \\ \phi(\left\|\nabla^m u\right\|^2)(-\Delta)^m & (-\Delta)^m \end{pmatrix}, \qquad (4.2)$$

$$F(U) = \begin{pmatrix} 0 \\ f(x) - g(u) \end{pmatrix}, \qquad (4.3)$$

$$D(H) = \{u \in H^{2m}(\Omega) \,|\, u \in L^2(\Omega), (-\Delta)^m u \in H^{2m}(\Omega)\} \times H^m.$$

We consider the usual graph norm in X, induced by the scale product,

$$(U, V)_X = (\phi \cdot \nabla^m u, \nabla^m \bar{y}) + (\bar{z}, v), \qquad (4.4)$$

with $U = (u, v), v = (y, z) \in X$, $\bar{y}, \bar{z}$ respectively denote the conjugation of y and x, $u, y \in H^{2m}(\Omega)$, $v, z \in H_0^m(\Omega)$, obviously, the operator H defined in (4.2) is monotone, for $U \in D(H)$,

$$\begin{aligned}
(HU, U)_X &= ((-v, \phi \cdot (-\Delta)^m u + (-\Delta)^m v), (u, v))_X \\
&= (-\phi \cdot \nabla^m v, \nabla^m \bar{u}) + (\bar{v}, \phi \cdot (-\Delta)^m u + (-\Delta)^m v) \\
&= (-\phi \cdot \nabla^m v, \nabla^m \bar{u}) + (\nabla^m \bar{v}, \phi \cdot \nabla^m u) + (\nabla^m \bar{v}, \nabla^m v) \\
&= \left\|\nabla^m v\right\|^2 \ge 0
\end{aligned} \qquad (4.5)$$

Therefore, $(HU, U)_X$ is a nonnegative and real number.

To determine the eigenvalues of H, we consider the following eigenvalue equation

$$HU = \lambda U, \quad U = (u, v) \in X \qquad (4.6)$$

That is

$$\begin{cases} -v = \lambda u, \\ \phi(\left\|\nabla^m u\right\|^2)(-\Delta)^m u + (-\Delta)^m v = \lambda v. \end{cases} \qquad (4.7)$$

The first equation of (4.7) is substituted into the second equation of (4.7), we obtain

$$\begin{cases} \lambda^2 u + \phi(\left\|\nabla^m u\right\|^2)(-\Delta)^m u - \lambda(-\Delta)^m u = 0, \\ u|_{\partial\Omega} = (-\Delta)^m u|_{\partial\Omega} = 0. \end{cases} \qquad (4.8)$$

Taking u inner product with the first equation of (4.8), we have

$$\lambda^2 \left\|u\right\|^2 + \phi(\left\|\nabla^m u\right\|^2)\left\|\nabla^m u\right\|^2 - \lambda\left\|\nabla^m u\right\|^2 = 0. \qquad (4.9)$$

(4.9) is considered an a yuan quadratic equation on λ, so we know

$$\lambda_k^{\pm} = \frac{\mu_k \pm \sqrt{\mu_k^2 - 4\mu_k \phi(\mu_k)}}{2} \qquad (4.10)$$

Where μ_k is the eigenvalue of $(-\Delta)^m$ in $H_0^m(\Omega)$, then $\mu_k = \lambda_1 k^{\frac{m}{n}}$. If $\mu_k \ge 4\phi(\mu_k)$, that is, $\mu_k \ge 4m_1$, the eigenvalues of H are all positive and real numbers, the corresponding eigenfunction have the form $U_k^{\pm}=(u_k,-\lambda_k^{\pm}u_k)$. For (4.10) and future reference, we note that for all $k\geq 1$,

$$\|\nabla^m u_k\|=\sqrt{\mu_k},\|u_k\|^2=1,\|\nabla^{-m}u_k\|=\frac{1}{\sqrt{\mu_k}}. \qquad (4.11)$$

Lemma 4.1 $g:H_0^m(\Omega)\to H_0^m(\Omega)$ is uniformly bounded and globally Lipschitz continuous.

Proof. $\forall u_1,u_2\in H_0^m(\Omega)$, we have

$$\|g(u_1)-g(u_2)\|=\|g'(\xi)(u_1-u_2)\|\leq|g'(\xi)|\|u_1-u_2\|.$$

with $\xi\in(u_1,u_2)$, because of (1), we can get

$$\|g(u_1)-g(u_2)\|=\|g'(\xi)(u_1-u_2)\|\leq c_5\|u_1-u_2\|.$$

Let $l=c_5$, then l is Lipschitz coefficient of $g(u)$.

Theorem 4.1 The following inequalities hold while $\mu_k\geq 4m_1$, if l is Lipschitz constant of $g(u)$, let $N_1\in N$ be so large such that $N\geq N_1$.

$$(\mu_{N+1}-\mu_N)(\tfrac{1}{2}-\tfrac{1}{2}\sqrt{2\mu_1-4m_1})\geq\frac{4l}{\sqrt{2\mu_1-4m_1}}+1. \qquad (4.12)$$

Then the operator H satisfies the spectral gap condition of (2.9).

Proof. When $\mu_k\geq 4m_1$, all the eigenvalues of H are real and positive, and we know that both sequences $\{\lambda_k^-\}_{k\geq1}$ and $\{\lambda_k^+\}_{k\geq1}$ are increasing.

The whole process of proof is divided into four steps.

(1) since $\lambda_k^{\pm}$ is arranged in nondecreasing order. According to Lemma 2.1, given N such that λ_N^- and λ_{N+1}^- are consecutive, we separate the eigenvalue of H as

$$\begin{aligned}\sigma_1&=\{\lambda_j^-,\lambda_k^+\,|\max\{\lambda_j^-,\lambda_k^+\}\leq\lambda_N^-\},\\ \sigma_2&=\{\lambda_j^-,\lambda_k^{\pm}\,|\lambda_j^-\leq\lambda_N^-\leq\min\{\lambda_j^-,\lambda_k^{\pm}\}\}.\end{aligned} \qquad (4.13)$$

(2) we will decomposition of X.

$$\begin{aligned}X_1&=span\{U_j^-,U_k^+\,|\lambda_j^-,\lambda_k^+\in\sigma_1\},\\ X_2&=span\{U_j^-,U_k^{\pm}\,|\lambda_j^-,\lambda_k^{\pm}\in\sigma_2\}.\end{aligned} \qquad (4.14)$$

Our purpose is made these two subspaces orthogonal and satisfies spectral inequality (2.9). $\Lambda_1=\lambda_N^-,\Lambda_2=\lambda_{N+1}^-$, we further decompose $X_2=X_c\oplus X_R$, with

$$\begin{aligned}X_c&=span\{U_j^-\,|\lambda_j^-\leq\lambda_N^-<\lambda_j^+\},\\ X_R&=span\{U_R^{\pm}\,|\lambda_N^-<\lambda_k^{\pm}\}.\end{aligned} \qquad (4.15)$$

and let $X_N=X_1\oplus X_c$. Next, we will stipulate a eigenvalue scale product of X such that X_1 and X_2 are orthogonal, so we need to introduce two functions

$\Phi:X_N\to R,\Psi:X_R\to R$.

$$\begin{aligned}\Phi(U,V)&=2(\nabla^m u,\nabla^m\bar{y})+2(\nabla^{-m}\bar{z},\nabla^m u)\\ &+2(\nabla^{-m}v,\nabla^m\bar{y})+4(\nabla^{-m}\bar{v},\nabla^{-m}v)\\ &-4\phi(\|\nabla^m u\|^2)(u,y)\end{aligned} \qquad (4.16)$$

$$\Psi(U,V)=(\nabla^m u,\nabla^m\bar{y})+(\nabla^{-m}\bar{v},\nabla^m u)-(\nabla^{-m}\bar{z},\nabla^m y) \qquad (4.17)$$

With $U=(u,v),V=(y,z),\bar{y},\bar{z}$ respectively denotes the conjugation of y,z.

Let $U=(u,v)\in X_N$, then

$$\begin{aligned}&\Phi(U,U)\\ &=2(\nabla^m u,\nabla^m\bar{u})+2(\nabla^{-m}\bar{v},\nabla^m u)+2(\nabla^{-m}v,\nabla^m\bar{u})\\ &+4(\nabla^{-m}\bar{v},\nabla^{-m}v)-4\phi(\|\nabla^m u\|^2)(u,u)\\ &\geq 2\|\nabla^m u\|^2-4\|\nabla^{-m}v\|^2-\|\nabla^m u\|^2+4\|\nabla^{-m}v\|^2-4\phi\|u\|^2\\ &=\|\nabla^m u\|^2-4\phi\|u\|^2\quad\geq(\mu_1-4m_1)\|u\|^2\end{aligned} \qquad (4.18)$$

Since. For any $k,\mu_k\geq 4m_1$, we can conclude that $\Phi(U,U)\geq0$. For all $U\in X_N$, analogously, for $U\in X_R$, we have

$$\begin{aligned}\Psi(U,U)&=(\nabla^m u,\nabla^m\bar{u})+(\nabla^{-m}\bar{v},\nabla^m u)\\ &-(\nabla^{-m}\bar{v},\nabla^m u)\quad\geq\mu_1\|u\|^2\geq0\end{aligned} \qquad (4.19)$$

So, we also find that $\Psi(U,U)\geq0$ for all $U\in X_R$. Therefore, we define a scale product with Φ and Ψ in X.

$$<<U,V>>_X=\Phi(P_NU,P_NV)+\Psi(P_RU,P_RV) \qquad (4.20)$$

Where P_N,P_R are respectively the projection: $X\to X_N$, $X\to X_R$, for brief, we can rewriter (4.20) as the following.

$$<<U,V>>_X=\Phi(U,V)+\Psi(U,V). \qquad (4.21)$$

We will show that these two subspace X_1,X_2

Defined in (4.14) are orthogonal in regard to the scale product (4.21) in the following process, in fact X_N and X_c are orthogonal, that is $<<U_j^+,U_j^->>_X=0$, for every $U_j^+\in X_c,U_j^-\in X_N$, we can compute from (4.16)

$$\begin{aligned}&<<U_j^+,U_j^->>_X=\Phi(U_j^+,U_j^-)\\ &=2(\nabla^m u_j,\nabla^m\bar{u}_j)-2\lambda_j^+(\nabla^{-m}\bar{u}_j,\nabla^m u_j)\\ &-2\lambda_j^-(\nabla^{-m}u_j,\nabla^m\bar{u}_j)+4\lambda_j^+\lambda_j^-(\nabla^{-m}\bar{u}_j,\nabla^{-m}u_j)\\ &-4\phi\|u_j\|^2\\ &=2\|\nabla^m u_j\|^2-2(\lambda_j^-+\lambda_j^+)\|u_j\|^2\\ &+4\lambda_j^+\lambda_j^-\|\nabla^{-m}u_j\|^2-4\phi\|u_j\|^2\\ &=2\mu_j-2(\lambda_j^-+\lambda_j^+)+4\lambda_j^+\lambda_j^-\cdot\frac{1}{\mu_j}-4\phi\end{aligned} \qquad (4.22)$$

According to (4.10), we have $\lambda_j^++\lambda_j^-=\mu_j,\lambda_j^+\lambda_j^-=\phi\mu_j$

So

$$<<U_j^+,U_j^->>_X=\Phi(U_j^+,U_j^-)=0 \qquad (4.23)$$

(3) Next, we estimate the Lipschitz constant l_F of F,

$F(U)=(0,f(x)-g(u))^T$ $g:H^m\to H^m$ is globally.

Lipschitz continuous with Lipschitz constant l, from (4.17), (4.18), for arbitrarily $U=(u,v)\in X$, we have

$$\begin{aligned}\|U\|_X^2 &= \Phi(P_1U, P_1U) + \Psi(P_2U, P_2U) \\ &\geq (\mu_1 - 4\phi)\|P_1u\|^2 + \mu_1\|P_2u\|^2 \\ &\geq (2\mu_1 - 4m_1)\|u\|^2\end{aligned} \tag{4.24}$$

Given $U = (u, v), V = (\hat{u}, \hat{v}) \in X$, we have

$$\begin{aligned}\|F(U) - F(V)\|_X &= \|g(u) - g(\hat{u})\| \\ &\leq l\|u - \hat{u}\| \\ &\leq \frac{l}{\sqrt{2\mu_1 - 4m_1}}\|U - V\|_X\end{aligned} \tag{4.25}$$

That we can claim that

$$l_F \leq \frac{l}{\sqrt{2\mu_1 - 4m_1}} \tag{4.26}$$

(4) Now, we need verify the spectral gap condition (2.9) holds.

Following the above mentioned $\Lambda_1 = \lambda_N^-$ and $\Lambda_2 = \lambda_{N+1}^-$, we can obtain

$$\begin{aligned}\Lambda_2 - \Lambda_1 &= \lambda_{N+1}^- - \lambda_N^- \\ &= \tfrac{1}{2}(\mu_{N+1} - \mu_N) + \tfrac{1}{2}(\sqrt{R(N)} - \sqrt{R(N+1)})\end{aligned} \tag{4.27}$$

Where $R(N) = \mu_N^2 - 4\phi\mu_N$.

We determine $N_1 > 0$ such that for all $N \geq N_1$, Let $R_1(N) = 1 - \sqrt{\frac{1}{2\mu_1 - 4m_1} - \frac{4m_1}{\mu_{N+1}(2\mu_1 - 4m_1)}}$, we can compute

$$\begin{aligned}&\sqrt{R(N)} - \sqrt{R(N+1)} + \sqrt{2\mu_1 - 4m_1}(\mu_{N+1} - \mu_N) \\ &= \sqrt{2\mu_1 - 4m_1}(\mu_{N+1}R_1(N+1) - \mu_N R_1(N))\end{aligned} \tag{4.28}$$

By the former assume $\varepsilon \leq m_0 \leq \varphi(s) \leq m_1 = \frac{2\mu_1 - 1}{4}$, we can easily know that

$$\lim_{N\to\infty}(\sqrt{R(N)} - \sqrt{R(N+1)} + \sqrt{2\mu_1 - 4m_1}(\mu_{N+1} - \mu_N)) = 0 \tag{4.29}$$

Then, combining (4.26) (4.27) (4.12) and (4.29), we obtain

$$\begin{aligned}\Lambda_2 - \Lambda_1 &> (\mu_{N+1} - \mu_N)(\tfrac{1}{2} - \tfrac{1}{2}\sqrt{2\mu_1 - 4m_1}) - 1 \\ &\geq \frac{4l}{\sqrt{2\mu_1 - 4m_1}} \geq 4l_F\end{aligned} \tag{4.30}$$

So, the prove is ended.

Theorem 4.2. Under the condition of Theorem 4.1, the initial boundary value problem (1.1)-(1.3) admits an inertial manifold μ in X of the form

$$\mu = graph(m) := \{\zeta + m(\zeta) : \zeta \in X_1\}, \tag{4.31}$$

where X_1, X_2 are as in (4.14) and $m : X_1 \to X_2$ is a Lipschitz continuous function.

V. Summary

In section 4, we have proved that the inertial manifold exists when $\mu_k \geq 4m_1$, next, we discuss the existence of the inertial manifold when $\mu_k < 4m_1$.

Since $\mu_k < 4m_1$, the eigenvalues of H are complex, with $\mathrm{Re}\,\lambda_k^{\pm} = \frac{1}{2}\mu_k$, and when N is sufficiently large, discuss the results with Theorem 4.1 is similar, so the proof procedure is omitted.

Acknowledgements

The authors express their sincere thanks to the anonymous reviewer for his/her careful reading of the paper, giving valuable comments and suggestions. These contributions greatly improved the paper.

Reference

[1] S.M. Zheng, A.Milani, Exponential attractors and inertial manifolds for singular perturbations of the Cahn-Hilliard equations, Nonlinear Anal. 57 (2004) 843-877.
[2] Jingzhu Wu, Guoguang Lin. An inertial manifold of the two-dimensional strongly damped Boussinesq equation. Journal of Yunnan University (Natural Science edition),Vol. 32(S2) 119-224.
[3] Z.J. Yang, Z.M. Liu. Exponential attractor for the Kirchhoff equations with strong nonlinear damping and supercritical nonlinearity. Journal of Applied Mathematics Letters 46(2015) 127-132.
[4] I. Chueshov, Long-time dynamics of Kirchhoff wave models with strong nonlinear damping, J. Differential Equations 252(2012) 1229-1262.
[5] Ke Li, Zhijian Yang. Exponential attractors for the strongly damped wave equation, Applied Mathematics and Computation 220(2013) 155-165.
[6] Meihua Yang, Chunyou Sun, Exponential attractors for the strongly damped wave equations, Nonlinear Analysis: Real World Applications 11(2010)913-919.
[7] Xu Guigui, Wang Libo, Lin Guoguang, Inertial manifolds for a class of the retarded nonlinear wave equations, Mathematica Applicata 2014, 27(4):887-891.
[8] R. Temam, Inertial manifolds , The Mathematical Intelligencer VOL. 12, NO. 4, 1990.
[9] Ahmed Y. Abdallah, Uniform exponential attractor for first order non-autonmous lattice dynamical systems, J. Differential Equations 251(2011)1489-1504.
[10] Grzegorz Lukaszewicz, On the existence of an exponential attractor for a planar shear flow with the Tresca friction condition, Nonlinear Analysis: Real World Applications 14(2013) 1585-1600.
[11] Ruijin Lou, Penghui Lv, Guoguang Lin. Exponential attractors and inertial manifolds for a class of generalized nonlinear Kirchhoff-Sine-Gordon equation. Journal of Advances in Mathematics. Volume 12 Number 06, ISSN 2347-1921.
[12] Ciprian Foias, Inertial manifolds for nonlinear evolutionary equations, Journal of Differential Equations 73, 309-353(1988).
[13] J.C. Robinson, Infinite dimensional dynamical systems, Cambridge University Press, London, 2001.
[14] Ling Chen, Wei Wang, Guoguang Lin. The global attractors and their Hausdorff and fractal dimensions estimation for the higher-order nonlinear Kirchhoff-type equation. Journal of Advances in Mathematics. Volume 12 Number 09, ISSN 2347-1921.
[15] Babin A V, Vishik MI. Regular attractors of semi-group and evolution equations [J]. Journal of Mathematics pure and Applications, 1983, 62(5):441-491.
[16] Shang Yadong, Guo boling, Exponential attractor for the generalized symmetric regularized long wave equation with damping term, Applied Mathematics and Mechanics, 2003,26(3):259-266.
[17] Eden A, Foias C, Nicolanenko B, et al. Exponential Attractors for Dissipative Evolution Equations [M]. New York: Masson, Paris, Wiely, 1994,36-48.
[18] R Tmam, Infinte dimensional dynamical systems in mechanics and physics, Springerverlag, New York, 1988.

Ling Chen, Department of Mathematics, Yunnan University, Kunming, Yunnan 650091, People's Republic of China
Wei Wang, Department of Mathematics, Yunnan University, Kunming, Yunnan 650091, People's Republic of China
Guoguang Lin, Department of Mathematics, Yunnan University, Kunming, Yunnan 650091, People's Republic of China