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Darboux transformation and rational solutions for the general Hirota equation

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***Abstract*— In this paper, we investigate the rational solutions of the general Hirota equation by the generalized Darboux transformation. Based on two kinds of seed solutions, the rogue wave solutions and lump solutions are derived in terms of rational forms. It is shown that this equation supports lump solutions with constant boundary condition.**

***Index Terms*— Darboux transformation, rational solutions, rogue wave solutions, lump solutions**

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I. Introduction

The Darboux transformation (DT) [1]-[3], originating from the work of Darboux in 1882 on the Sturm-Liouville equation, is a powerful method for constructing exact solutions of nonlinear wave equations [4]-[9]. The Darboux theory is presented in several monographs and review papers [1, 2]. In the literature, various approaches have been proposed to find a DT for a given equation, for instance, the operator factorization method [10], the gauge transformation method [11], and the loop group transformation [12]. It is remarked that the DT is very efficient for construction of rouge wave solutions, which have gained more and more interests in the study of physics. In this paper, we study the general Hirota equation

$$u_t = i\,\alpha\left(u_{xx} + 2\left|u\right|^2 u + \omega\,u\right) + \epsilon\left(u_{xxx} + 6\left|u\right|^2 u_x\right), \quad (1)$$

where is the complex amplitude of the pulse envelope, the parameter ϵ denotes the relative width of the spectrum that arises due to the quasi-monochromocity, and ω is a constant. The aim of this paper is to construct the rational solutions of Eq. (1) by DT, including the rogue wave solutions and lump solutions.

This paper is organized as follows. In Section 2, the *N*-step Darboux transformation and generalized Darboux transformation for the general Hirota equation are proposed. In Section 3, some rational rogue wave and lump solutions of the general Hirota equation are derived and their figures are plotted. The main results of the paper are summarized in Section 4.

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II. The generalized Darboux transformation

The Lax pair (i.e. linear spectral problem) [13] of the general Hirota equation (1) is

$$\varphi_x = U\,\varphi, \qquad \varphi_t = V\,\varphi, \quad (2)$$

where the matrices *U* and *V* have the forms

$$U = \begin{pmatrix} -i\lambda & u \\ -u^* & i\lambda \end{pmatrix}, \quad V = \begin{pmatrix} 4\epsilon i\lambda^3 - 2i\alpha\lambda^2 - 2i\lambda\epsilon|u|^2 + a_{11} & -4\epsilon\lambda^2 u + 2(\alpha u - i\epsilon\lambda u_x) + a_{12} \\ 4\epsilon\lambda^2 u^* - 2\lambda(\alpha u^* - i\epsilon u_x^*) + a_{21} & -4\epsilon i\lambda^3 + 2i\alpha\lambda^2 + 2i\lambda\epsilon|u|^2 - a_{11} \end{pmatrix}, \quad (3)$$

where $\varphi = (f, g)^T$ (here *T* denotes the transpose) , λ is the spectral parameter, $a_{12} = \epsilon\,u_{xxx} + i\,\alpha\,u_x + 2\,\epsilon\,|u|^2\,u$ and

$a_{11} = \epsilon\,u_x\,u^* - \epsilon\,u_x^*\,u + i\,\alpha\,|u|^2 + \frac{1}{2}\,i\,\alpha\,\omega,$

$a_{21} = -\epsilon\,u_{xx}^* + i\,\alpha\,u_x^* - 2\,\epsilon\,|u|^2\,u^*.$

DT technique is a method which can derive exact solutions from trivial seed solutions in a purely algebraic procedure for the integrable nonlinear wave equations [14]-[18]. Main feature of DT is that the Lax pair associated with the nonlinear wave equations remains covariant under the gauge transformation. For the Lax pair (2) with (3) , we first construct the gauge transformation

$$\varphi^{(1)} = T^{(1)}\,\varphi = (\lambda\,I - S)\,\varphi, \quad (4)$$

where I denotes the identity matrix and $S = (s_{ij}^{(0)})_{2\times 2}$ is a 2×2matrix. Further, if matrix *S* takes the following form

$$S = H\,\Lambda\,H^{-1}, \quad \text{with} \quad \Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1^* \end{pmatrix}, \quad H = \begin{pmatrix} f_1 & -g_1^* \\ g_1 & f_1^* \end{pmatrix}, \quad (5)$$

It can be verified that if $(f_1, g_1)^T$ is a solution of Lax pair (2) with $\lambda = \lambda_1$, then$(-g_1^*, f_1^*)^T$ is also the solution of Lax pair(2) corresponding to$\lambda = \lambda_1^*$. Thus under the gauge transformation $\varphi^{(1)} = T^{(1)}\,\varphi$, the Lax pair (2) becomes

$$\varphi_x^{(1)} = U^{(1)}\,\varphi^{(1)}, \qquad \varphi_t^{(1)} = V^{(1)}\,\varphi^{(1)}, \quad (6)$$

where $U^{(1)}$ and $V^{(1)}$ have the same forms as U and V in Eq (3) except replacing u with $u^{(1)}$.

Through the gauge transformation (4) and Eqs. (5)-(6), the 1-step DT of the Eq. (1) is found as

$$\varphi^{(1)} = T^{(1)}\,\varphi, \qquad u^{(1)} = u - 2\,i\,s_{21}^{(0)}, \tag{7}$$

where

$$T^{(1)} = \lambda\,I - S = \begin{pmatrix} \lambda - s_{11}^{(0)} & -s_{12}^{(0)} \\ -s_{21}^{(0)} & \lambda - s_{22}^{(0)} \end{pmatrix}, \qquad s_{21}^{(0)} = \frac{(\lambda_1 - \lambda_1^*) f_1^*\, g_1}{|f_1|^2 + |g_1|^2}. \tag{8}$$

Furthermore, let $\varphi_2 = (f_2, g_2)^T$ be a solution of Lax pair (2) at $\lambda = \lambda_2$. Employing the map $\varphi_2^{(1)} = T^{(1)}|_{\lambda=\lambda_2}\varphi_2$ with $\varphi_2^{(1)} = (f_2^{(1)}, g_2^{(1)})^T$, we have the 2-step DT of the Eq. (1) as

$$\varphi^{(2)} = T^{(2)} T^{(1)}\,\varphi, \quad u^{(2)} = u^{(1)} - 2\,i\,s_{21}^{(1)} = u - 2\,i\sum_{j=0}^{1} s_{21}^{(j)}, \tag{9}$$

where

$$T^{(2)} = \begin{pmatrix} \lambda - s_{11}^{(1)} & -s_{12}^{(1)} \\ -s_{21}^{(1)} & \lambda - s_{22}^{(1)} \end{pmatrix}, \qquad s_{21}^{(1)} = \frac{(\lambda_2 - \lambda_2^*) f_2^{(1)*}\, g_2^{(1)}}{|f_2^{(1)}|^2 + |g_2^{(1)}|^2}. \tag{10}$$

The 2-step DT may be iterated sequentially. In the general case, we have the following theorem:

Theorem 1. Let $\varphi_k\ (k = 1, 2, \cdots, N)$ be N distinct solutions of the Eq. (2) at λ_k, then the N-step DT of Eq. (1) is

$$\varphi^{(n)} = T^{(n)}\,T^{(n-1)}\,\cdots\,T^{(1)}\,\varphi, \qquad u^{(n)} = u - 2\,i\sum_{j=0}^{n-1} s_{21}^{(j)}, \tag{11}$$

where $\varphi_{j+1}^{(j)} = (f_{j+1}^{(j)}, g_{j+1}^{(j)})^T = (T^{(j)} T^{(j-1)} \cdots T^{(1)})|_{\lambda=\lambda_{j+1}}\varphi_{j+1}$ and

$$T^{(j+1)} = \begin{pmatrix} \lambda - s_{11}^{(j)} & -s_{12}^{(j)} \\ -s_{21}^{(j)} & \lambda - s_{22}^{(j)} \end{pmatrix}, \qquad s_{21}^{(j)} = \frac{(\lambda_{j+1} - \lambda_{j+1}^*) f_{j+1}^{(j)\,*}\, g_{j+1}^{(j)}}{|f_{j+1}^{(j)}|^2 + |g_{j+1}^{(j)}|^2}. \tag{12}$$

Next, we begin to search for a generalized DT of the general Hirota equation (1). Suppose that $\varphi_2 = \varphi_1\,(\lambda_1 + \sigma)$ is a special solution for Eq. (2), then after transformation we have $\varphi_2^{(1)} = T_1^{(1)}\,\varphi_2$. Expanding φ_2 at λ_1, we have [19]

$$\varphi_1\,(\lambda_1 + \sigma) = \varphi_1^{[0]} + \varphi_1^{[1]}\,\sigma + \varphi_1^{[2]}\,\sigma^2 + \cdots + \varphi_1^{[N]}\,\sigma^N + \cdots, \tag{13}$$

where $\varphi_1^{[0]} = \varphi_1,\ \varphi_1^{[k]} = \frac{1}{k!}\frac{\partial^k}{\partial\lambda^k}\varphi_1\,(\lambda)|_{\lambda=\lambda_1}$ and $\sigma\sigma$ is a small parameter. Through the limit process

$$\lim_{\sigma\to 0}\frac{[T_1^{(1)}|_{\lambda=\lambda_1+\sigma}]\,\varphi_2}{\sigma} = \lim_{\sigma\to 0}\frac{[\sigma + T_1^{(1)}|_{\lambda=\lambda_1}]\,\varphi_2}{\sigma} = \varphi_1 + T_1^{(1)}|_{\lambda=\lambda_1}\,\varphi_1^{[1]} \equiv \varphi_1^{(1)}, \tag{14}$$

we can find another solution to the Eq. (2) with $u^{(1)}$ and spectral parameter $\lambda = \lambda_1$. Continuing the above process and combining all the DT, a generalized DT is constructed. Thus we have the following theorem:

Theorem 2. Le $\varphi_1, \varphi_2, \cdots, \varphi_N$ be N distinct solutions of the Eq. (2) at $\lambda_1, \cdots, \lambda_N$, respectively, and

$$\varphi_i\,(\lambda_i + \sigma) = \varphi_i^{[0]} + \varphi_i^{[1]}\,\sigma + \varphi_i^{[2]}\,\sigma^2 + \cdots + \varphi_i^{[N]}\,\sigma^N + \cdots (i = 1, 2, \cdots, n), \tag{15}$$

be their expansions, where $\varphi_i^{[0]} = \varphi_i, \varphi_i^{[j]} = \frac{1}{j!}\frac{\partial^j}{\partial\lambda^j}\varphi_i\,(\lambda)|_{\lambda=\lambda_i}\quad (j = 1, 2, \cdots)$.

Defining $T = \Gamma_n\,\Gamma_{n-1}\,\cdots\,\Gamma_1\,\Gamma_0,\quad \Gamma_k = T_k^{(m_k)}\,\cdots\,T_k^{(1)}(i \geq 1),\quad \Gamma_0 = I$, where

$$T_k^{(j+1)} = \begin{pmatrix} \lambda_k - s_{11}^{(j)} & -s_{12}^{(j)} \\ -s_{21}^{(j)} & \lambda_k - s_{22}^{(j)} \end{pmatrix}, \tag{16}$$

$$\varphi_k^{(j)} = \lim_{\sigma\to 0}\frac{[\sigma + T_k^{(j)}|_{\lambda=\lambda_k}]\cdots[\sigma + T_k^{(2)}|_{\lambda=\lambda_k}]\,[\sigma + T_k^{(1)}|_{\lambda=\lambda_k}]\,\Gamma_{k-1}\,(\lambda_k + \sigma)\cdots\Gamma_1\,(\lambda_k + \sigma)\,\Gamma_0\,\varphi_k\,(\lambda_k + \sigma)}{\sigma^j}, \tag{17}$$

$(1 \leq j < m_i)$, then the transformations

$$\varphi^{(N)} = T\,\varphi, \qquad u^{(N)} = u + 2\,i\sum_{j=0}^{n-1}\sum_{k=1}^{m_i} s_{21}^{(j)}\,[m_k], \qquad (N = n - 1 + \sum_{k=1}^{n-1} m_k), \tag{18}$$

are the generalized DT for the general Hirota equation (1).

III. RATIONAL SOLUTIONS

Let us consider an example to illustrate the application of Theorem 2 to construct rational solutions for the general Hirota equation (1). In doing so, starting with the seed solution $u = c\, e^{i\,\alpha\,(\omega+2\,c^2)\,t}$. Without loss of generality, assuming that $c = 1$, the solution of the Lax pair (2) at$\lambda = i\,h$ is

$$\varphi_1\,(f) = \left(\begin{array}{c} e^{B}\,(C_2\,e^{A} + C_1\,e^{-A}) \\ i\,e^{-B}\,(C_1\,e^{A} - C_2\,e^{-A}) \end{array}\right), \tag{19}$$

where$C_1 = \sqrt{\sqrt{1+\lambda^2}+\lambda}, C_2 = \sqrt{\sqrt{1+\lambda^2}-\lambda}, B = \frac{1}{2}\,i\,\alpha\,(2+\omega)\,t,$
$A = i\,\sqrt{1+\lambda^2}\,[x + 2\,(\epsilon + \lambda\,\alpha - 2\,\lambda^2\,\epsilon)\,t + (b_1 + i\,d_1)\,\sigma^2 + (b_2 + i\,d_2)\,\sigma^4 + (b_3 + i\,d_3)\,\sigma^6 + ... + (b_N + i\,d_N)\,\sigma^{2N}],$
where $b_k, d_k (k = 1, 2, ..., N)$ are real free parameters.

Let $h = 1 + \sigma^2$, expanding the vector function $\varphi_1\,(\sigma)$ at $\sigma = 0$, we have

$$\varphi_1\,(\sigma) = \varphi_1^{[0]} + \varphi_1^{[1]}\sigma^2 + \varphi_1^{[2]}\,\sigma^4 + \varphi_1^{[3]}\,\sigma^6 + \cdots, \tag{20}$$

where $\varphi_1^{[0]} = \left(\begin{array}{c} P^{[0]} \\ Q^{[0]} \end{array}\right), \varphi_1^{[1]} = \left(\begin{array}{c} P^{[1]} \\ Q^{[1]} \end{array}\right), \varphi_1^{[2]} = \left(\begin{array}{c} P^{[2]} \\ Q^{[2]} \end{array}\right)$, and $(P^{[i-1]}, Q^{[i-1]})^T, (i = 1, 2, 3)$are given in Appendix A.

It is clear that $\varphi_1^{[0]}$is a solution for Eq. (2) at $\lambda = i$. In what follows, we impose $\alpha = 1, \epsilon = 0.1$, and only discuss three cases: *N* = 1, *N* = 2 and *N* = 3.

(1). When *N* = 1, by means of formula (14), we have

$$\varphi_1^{(1)} = \lim_{f\to 0} \frac{[f^2 + T_1^{(1)}]\,\varphi_1\,(\sigma)}{\sigma^2} = T_1^{(1)}\,\varphi_1^{[1]} + \varphi_1^{[0]}. \tag{21}$$

From Eqs. (19) and (20),the first-order rogue wave solution is derived as

$$u_{rw}^{(1)} = -\frac{e^{i\,t}\,(75 - 100\,x^2 - 120\,x\,t - 436\,t^2 + 400\,i\,t)}{25 + 100\,x^2 + 120\,x\,t + 436\,t^2}, \tag{22}$$

where we have used the notation$u_{rw}^{(N)}$ represent the N-order rogue wave solution for simplicity.

(2). When N = 2, by means of the formula of $\varphi_1^{(2)}$1in Eq. (17), we obtain

$$\varphi_1^{(2)} = \lim_{\sigma\to 0} \frac{[\sigma^2 + T_1^{(2)}]\,[\sigma^2 + T_1^{(1)}]\,\varphi_1\,(\sigma)}{\sigma^4} = T_1^{(2)}\,T_1^{(1)}\,\varphi_1^{[2]} + (T_1^{(1)} + T_1^{(2)})\,\varphi_1^{[1]} + \varphi_1^{[0]}, \tag{23}$$

which helps to derive the second-order rogue wave solution as

$$u_{rw}^{(2)} = \frac{M_{2\,j}}{N_{2\,j}}, \tag{24}$$

where for$j = 1$, we have$u_{rw}^{(2)}$with parameters$b_1 = 0, d_1 = 0,$ and for $j = 2$, we have $u_{rw}^{(2)}$with parameters $b_1 = 100, d_1 = 0.$ Here $M_{2\,j}$ and $N_{2\,j}$ $(j = 1, 2)$ are listed in Appendix B.

(3). When *N* = 3, by means of the formula in Eq. (17) the third-order rogue wave solutions can also be obtained, which is omitted since it is very complicated. Thus we only show the figures of $u_{rw}^{(3)}$ for simplicity. The first-order, second-order and third-order rogue wave solutions are displayed in Figure 1 and Figure 2, respectively.

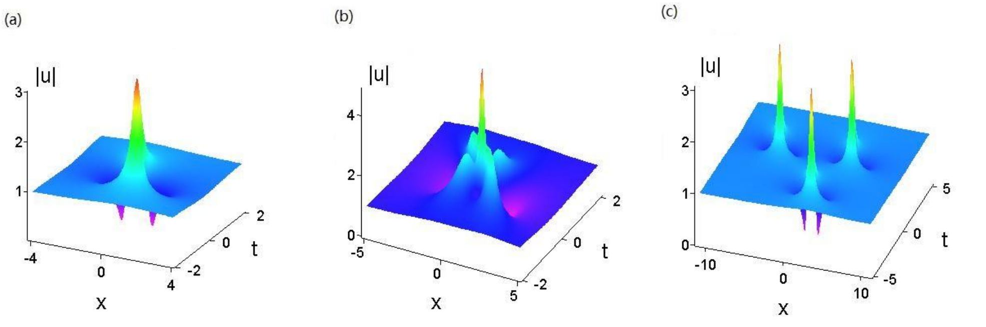

Figure 1: (a) The first-order rogue wave$u_{rw}^{(1)}$. (b) The second-order rogue wave$u_{rw}^{(2)}$ with parameters $b_1 = d_1 = 0$. (c) The second-order rogue wave $u_{rw}^{(2)}$ with parameters$b_1 = 100, d_1 = 0$. The other parameters are $\alpha = 1, \epsilon = 0.1, \omega = -1, \lambda = i$.

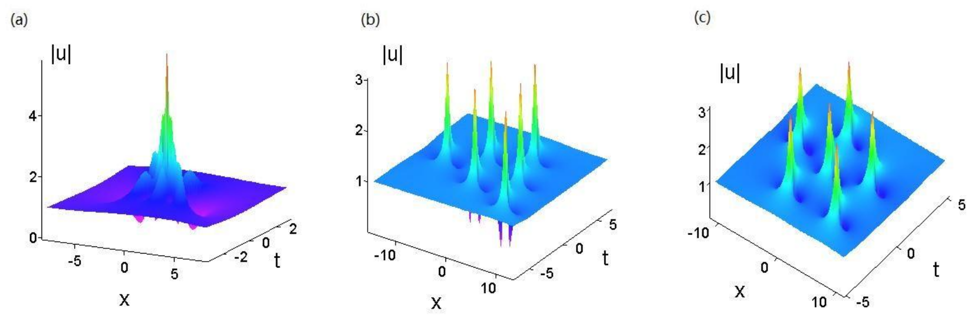

Figure 2: Plots of the third-order rogue wave $u_{rw}^{(3)}$. (a) Parameters $b_1 = d_1 = b_2 = d_2 = 0$. (b) Parameters $b_1 = 100, d_1 = b_2 = d_2 = 0$. (c) Parameters $b_1 = d_1 = b_2 = 0, d_2 = 1000$. The other parameters are $\alpha = 1, \epsilon = 0.1, \omega = -1, \lambda = i$.

Finally, let us provide another example to illustrate the application of Theorem 2. To do so, starting with the constant seed solution $u = \sqrt{-\frac{\omega}{2}}$ $(\omega < 0)$, the solution of the Lax pair (2) at $\lambda = i\,h$ is

$$\varphi_1(f) = \begin{pmatrix} C_2\, e^{A} + C_1\, e^{-A} \\ i\,(C_1\, e^{A} - C_2\, e^{-A}) \end{pmatrix}. \quad (25)$$

Also we can obtain the rogue wave solutions of the general Hirota equation (1) by using the same procedure as above. Although the seed solution $u = \sqrt{-\frac{\omega}{2}}$ $(\omega < 0)$ is a real constant instead of plane wave, we can also obtain the rational rogue wave solutions, which seems an interesting result: in other literature the rogue wave solution can usually be obtained starting from a plane wave seed solution, however, we can derive the rogue wave solutions from a constant seed solution in this paper. The most interesting result is that we can utilize the Taylor expansion and limit procedure to obtain the lump solutions from the constant seed solution. In the following, we impose parameters $\alpha = 0, \epsilon = 0.1, \omega = -1$ and only consider three cases, i.e. $N = 1$, $N = 2$ and $N = 3$.

(1). When $N = 1$, the first-order lump solution is derived as

$$u_{ls}^{(1)} = \frac{-75 + 100\,x^2 + 120\,x\,t + 36\,t^2}{25 + 100\,x^2 + 120\,x\,t + 36\,t^2}, \quad (26)$$

which is a real rational solution compared with the complex rational rogue wave solution in (22), where we have used the notation $u_{ls}^{(N)}$ to represent the N-order lump solution for simplicity.

(2). When $N = 2$, we can obtain the second-order lump solutions as

$$u_{ls}^{(2)} = \frac{X_{2\,j}}{Y_{2\,j}},$$

where for $j = 1$, we have $u_{ls}^{(2)}$ with parameters $b_1 = 0, d_1 = 0$, and for $j = 2$, we have $u_{ls}^{(2)}$ with parameters $b_1 = 100, d_1 = 0$. Here $X_{2\,j}$ and $Y_{2\,j}$ $(j = 1, 2)$ are given in Appendix C. The first-order and second-order lump solutions are shown in Figure 3, which are localized in space and evolution in time.

(3). When $N = 3$, we can derive the third-order lump solution $u_{ls}^{(3)}$ which is too complicated to be written down here, so we only show their plots in Figure 4.

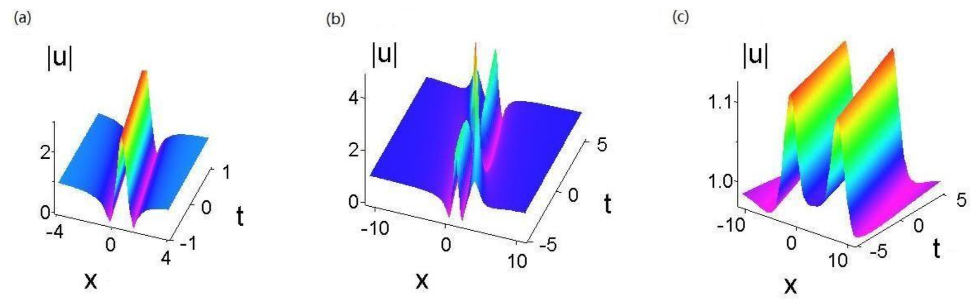

Figure 3: (a) The first-order lump solution $u_{ls}^{(1)}$. (b) The second-order lump solution $u_{ls}^{(2)}$ with parameters $b_1 = 0$ and $d_1 = 0$. (c) The second-order lump solution $u_{ls}^{(2)}$ with parameters $b_1 = 100$ and $d_1 = 0$. The other parameters are $\alpha = 0, \epsilon = 0.1, \omega = -1, \lambda = i$.

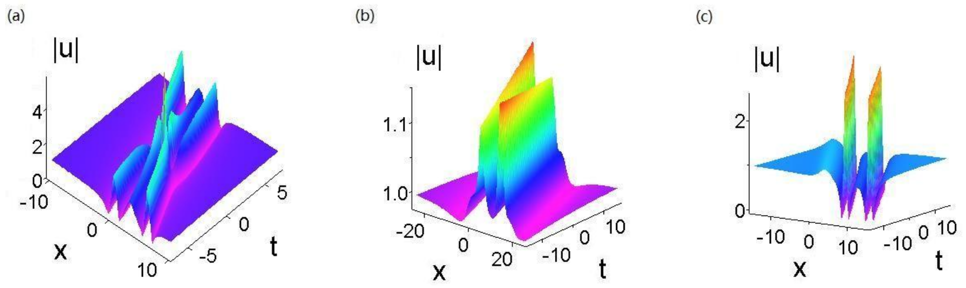

Figure 4: Plots of the third-order lump solution $u_{ls}^{(3)}$ under different parameters. (a) $b_1 = d_1 = b_2 = d_2 = 0$. (b) $b_1 = 100, d_1 = b_2 = d_2 = 0$. (c) $b_1 = d_1 = d_2 = 0, b_2 = 1000$. The other parameters are $\alpha = 0, \epsilon = 0.1, \omega = -1, \lambda = i$.

IV. Conclusions

In summary, we have obtained two kinds of rational solutions of the general Hirota equation by the generalized Darboux transformation. By means of the Taylor expansion and limit procedure [19], the generalized Darboux transformation is given from the *N*-step Darboux transformation. Then the one, two and three-order rational rogue wave solutions and lump solutions are proposed by choosing the plane wave seed solution and constant seed solution, respectively. The plots of these rational solutions are also provided. We show in this paper that complex nonlinear wave equation can support real lump solutions with pure constant boundary condition. Our results may be helpful to observe the evolution of rogue waves and lumps in a complicated optical system with high-order dispersion dispersion term like the general Hirota equation.

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Appendices.

Appendix A.

$Q^{[0]} = -\sqrt{2}, \quad P^{[0]} = \sqrt{2}, \quad Q^{[1]} = \frac{\sqrt{2}\,i}{100}\,(120\,x\,t + 400\,i\,x\,t + 25 - 60\,t - 100\,x - 200\,i\,t - 364\,t^2 + 240\,i\,t^2 + 100\,x^2), \quad P^{[1]} = \sqrt{2}\,(-i\,x + \frac{91}{25}\,i\,t^2 - \frac{3}{5}\,i\,t - i\,x^2 + \frac{12}{5}\,t^2 + 2\,t - \frac{6}{5}\,i\,x\,t + 4\,x\,t - \frac{1}{4}\,i), \quad Q^{[2]} = -\frac{\sqrt{2}}{6000}\,(1875 - 420000\,i\,x\,t + 15000\,x + 120000\,i\,x^2\,t + 144000\,i\,x\,t^2 - 116800\,i\,t^3 - 144000\,i\,x^2\,t^2 - 120000\,i\,x\,b_1 + 240000\,i\,t\,d_1 + 233600\,i\,x\,t^3 + 57000\,t - 45000\,x^2 - 150000\,x\,t + 586200\,t^2 + 36000\,x^2\,t - 218400\,x\,t^2 + 20000\,x^3 - 139680\,t^3 + 240000\,t\,b_1 - 24000\,x^3\,t + 218400\,x^2\,t^2 + 279360\,x\,t^3 + 120000\,d_1\,x + 72000\,d_1\,t - 10000\,x^4 - 74896\,t^4 + 60000\,i\,b_1 - 444000\,i\,t^2 - 72000\,i\,t\,b_1 - 60000\,d_1 + 150000\,i\,t + 174720\,i\,t^4 - 80000\,i\,x^3\,t),$

$P^{[2]} = \sqrt{2}\,(-\frac{5}{2}\,x\,t + d_1 - \frac{4681}{3750}\,t^4 - \frac{1}{3}\,x^3 - \frac{1}{4}\,x - 2\,i\,x^2\,t + 4\,i\,t\,d_1 + \frac{291}{125}\,t^3 - \frac{12}{5}\,i\,x\,t^2 - \frac{6}{5}\,i\,t\,b_1 + 4\,t\,b_1 + 2\,d_1\,x + \frac{582}{125}\,x\,t^3 + \frac{91}{25}\,x^2\,t^2 - \frac{37}{5}\,i\,t^2 + \frac{6}{5}\,d_1\,t + \frac{91}{25}\,x\,t^2 - \frac{3}{5}\,x^2\,t - i\,b_1 + \frac{1}{32} - \frac{5}{2}\,i\,t - \frac{19}{20}\,t - \frac{4}{3}\,i\,x^3\,t - \frac{2}{5}\,x^3\,t + \frac{364}{125}\,i\,t^4 - 7\,i\,x\,t + \frac{146}{75}\,i\,t^3 - 2\,i\,x\,b_1 - \frac{3}{4}\,x^2 + \frac{977}{100}\,t^2 - \frac{12}{5}\,i\,x^2\,t^2 + \frac{292}{75}\,i\,x\,t^3 - \frac{1}{6}\,x^4)$

Appendix B

$M_{21} = (-703125 + 34800000\,i\,t^3 + 7200000\,i\,x\,t^2 - 11250000\,i\,t - 18000000\,i\,x^2\,t + 12000000\,i\,t\,x^4 + 28800000\,i\,t^2\,x^3 + 121920000\,i\,t^3\,x^2 + 228115200\,i\,t^5 + 125568000\,i\,t^4\,x + 2812500\,x^2 + 8775000\,x\,t + 32062500\,t^2 + 83736000\,t^3\,x + 99180000\,x^2\,t^2 + 7800000\,x^3\,t + 147930000\,t^4 + 2250000\,x^4 - 33120000\,t^3\,x^3 - 75864000\,t^4\,x^2 - 68434560\,t^5\,x - 17400000\,x^4\,t^2 - 3600000\,x^5\,t - 82881856\,t^6 - 1000000\,x^6)\,e^{i\,t}, \quad N_{21} = 82881856\,t^6 + 68434560\,t^5\,x + 118378800\,t^4 + 75864000\,t^4\,x^2 + 33120000\,t^3\,x^3 + 5256000\,t^3\,x + 27877500\,t^2 + 17400000\,x^4\,t^2 - 20700000\,x^2\,t^2 - 600000\,x^3\,t + 3600000\,x^5\,t + 3825000\,x\,t + 1687500\,x^2 + 1000000\,x^6 + 140625 + 750000\,x^4, \quad M_{22} = (450000000\,t - 1752000000\,t^3 + 2812500\,x^2 + 8775000\,x\,t + 32062500\,t^2 + 1800000000\,x^2\,t + 2160000000\,x\,t^2 + 2250000\,x^4 + 147930000\,t^4 + 7800000\,x^3\,t + 83736000\,t^3\,x + 99180000\,x^2\,t^2 - 3600000\,x^5\,t - 17400000\,x^4\,t^2 - 75864000\,t^4\,x^2 - 68434560\,t^5\,x - 33120000\,x^3\,t^3 - 1000000\,x^6 - 82881856\,t^6 + 12000000\,i\,x^4\,t + 121920000\,i\,x^2\,t^3 - 22500703125 - 1080000000\,i\,x\,t + 28800000\,i\,x^3\,t^2 + 7200000\,i\,t^2\,x + 125568000\,i\,t^4\,x - 11250000\,i\,t + 3276000000\,i\,t^2 + 34800000\,i\,t^3 - 18000000\,i\,x^2\,t + 228115200\,i\,t^5 - 900000000\,i\,x^2 - 225000000\,i)\,e^{i\,t}, \quad N_{22} = 82881856\,t^6 + 68434560\,t^5\,x + 118378800\,t^4 + 75864000\,t^4\,x^2 + 1752000000\,t^3 + 5256000\,t^3\,x + 33120000\,x^3\,t^3 + 17400000\,x^4\,t^2 - 20700000\,x^2\,t^2 + 27877500\,t^2 - 2160000000\,x\,t^2 + 3825000\,x\,t - 1800000000\,x^2\,t - 600000\,x^3\,t + 3600000\,x^5\,t + 1350000000\,t + 22500140625 + 750000\,x^4 + 1687500\,x^2 + 1000000\,x^6$

Appendix C

$X_{21} = -(46656\,t^6 + 466560\,t^5\,x + 1944000\,t^4\,x^2 - 810000\,t^4 - 4536000\,x\,t^3 + 4320000\,x^3\,t^3 - 2812500\,t^2 + 324000000\,i\,t^2 - 9180000\,x^2\,t^2 + 5400000\,x^4\,t^2 + 900000000\,i\,x^2 - 8775000\,x\,t + 3600000\,x^5\,t - 7800000\,x^3\,t - 2812500\,x^2 + 1080000000\,i\,x\,t - 2250000\,x^4 + 1000000\,x^6 + 22500703125 + 225000000\,i)$, $Y_{21} = 46656\,t^6 + 466560\,t^5\,x - 421200\,t^4 + 1944000\,t^4\,x^2 - 1944000\,x\,t^3 + 4320000\,x^3\,t^3 + 5400000\,x^4\,t^2 - 2700000\,x^2\,t^2 + 3127500\,t^2 + 3825000\,x\,t - 600000\,x^3\,t + 3600000\,x^5\,t + 22500140625 + 1687500\,x^2 + 1000000\,x^6 + 750000\,x^4$, $X_{22} = -(46656\,t^6 + 466560\,t^5\,x - 810000\,t^4 + 1944000\,x^2\,t^4 - 4536000\,x\,t^3 + 4320000\,x^3\,t^3 - 9180000\,x^2\,t^2 + 5400000\,x^4\,t^2 - 2812500\,t^2 - 8775000\,x\,t + 3600000\,x^5\,t - 7800000\,x^3\,t - 2812500\,x^2 - 2250000\,x^4 + 1000000\,x^6 + 703125)$, $Y_{22} = 46656\,t^6 + 466560\,t^5\,x - 421200\,t^4 + 1944000\,x^2\,t^4 - 1944000\,x\,t^3 + 4320000\,x^3\,t^3 - 2700000\,x^2\,t^2 + 5400000\,x^4\,t^2 + 3127500\,t^2 + 3825000\,x\,t - 600000\,x^3\,t + 3600000\,x^5\,t + 140625 + 1687500\,x^2 + 750000\,x^4 + 1000000\,x^6$

References

[1]Matveev V B, Salle M A, *Darboux Transformation and Solitons* (Berlin, Springer-Verlag) 1991.

[2]Cieslinski J L *J, Phys. A: Math. Gener* 2009. 42 404003.

[3]Nimmo J J C, *Inverse Problems* 1992 8 219.

[4]Ablowitz M J, Kaup D J, Newell A C and Segur H, *Studies Appl. Math.* 1974 53 249.

[5]Kaup D J and Newell A C, *J. Math. Phys.* 1978 19 798.

[6]Hirota R, 7 *J. Phys. Soc. Japan* 199 66 2530.

[7]Wazwaz A M, *Can. J. Phys.* 2009 87 1227.

[8]Wazwaz A M, *Phys. Scr.* 2012 86 045501.

[9]Wazwaz A M, *Appl. Math. Model.* 2014 38 110-118.

[10] Adler M and Moser J, *Commun. Math. Phys.* 1978 61 1.

[11] Neugebauer G and Meinel R, *Phys. Lett. A* 1984 100 467-470.

[12] Terng C L and Uhlenbeck K *Commun. Pure Appl. Math.* ,2000 53 1.

[13] Wang D S, Optical Solitons in a Nonlinear Fiber Medium with Higher-Order Effects, in *Recent Progress in Optical Fiber Research*, Edited by Yasin M, Harun S W and Arof H, 2012 pp. 49-64.

[14] Gu C H, He H S and Zhou Z X, *Darboux Transformation in Soliton Theory and its Geometric Applications* (Shanghai Scientific and Technical Publishers) 2005.

[15] Ma W X, *Lett. Math. Phys.* 1997 39 33-49.

[16] Tao Y S and He J , *Phys. Rev. E* 2012 85 026601.

[17] Lu X, *Chaos* 2013 23 033137.

[18] Zhang Y, Nie X J and Zhaqilao, *Phys. Lett. A* 2014 378 191-197.

[19] Guo B L, Ling L M and Liu Q P, *Phys. Rev. E* 2012 85 026607.

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