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On a Scale Invariant Model of Statistical Mechanics, Kinetic Theory of Ideal Gas, and Riemann Hypothesis

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Abstract— **A scale invariant model of statistical mechanics is applied to derive invariant forms of conservation equations. A modified form of *Cauchy* stress tensor for fluid is presented that leads to modified *Stokes* assumption thus a finite coefficient of bulk viscosity. The phenomenon of *Brownian* motion is described as the state of equilibrium between suspended particles and molecular clusters that themselves possess *Brownian* motion. Physical space or *Casimir* vacuum is identified as a tachyonic fluid that is "stochastic ether" of *Dirac* or "hidden thermostat" of *de Broglie*, and is compressible in accordance with *Planck's* compressible ether. The stochastic definitions of *Planck* h and *Boltzmann* k constants are shown to respectively relate to the spatial and the temporal aspects of vacuum fluctuations. Hence, a modified definition of thermodynamic temperature is introduced that leads to predicted velocity of sound in agreement with observations. Also, a modified value of *Joule-Mayer* mechanical equivalent of heat is identified as the universal gas constant and is called *De Pretto* number 8338 which occurred in his mass-energy equivalence equation. Applying *Boltzmann's* combinatoric methods, invariant forms of *Boltzmann*, *Planck*, and *Maxwell-Boltzmann* distribution functions for equilibrium statistical fields including that of isotropic stationary turbulence are derived. The latter is shown to lead to the definitions of (electron, photon, neutrino) as the most-probable equilibrium sizes of (photon, neutrino, tachyon) clusters, respectively. The physical basis for the coincidence of normalized spacing between zeros of *Riemann* zeta function and the normalized *Maxwell-Boltzmann* distribution and its connections to *Riemann Hypothesis* are examined. The zeros of *Riemann* zeta function are related to the zeros of particle velocities or "stationary states" through *Euler's* golden key thus providing a physical explanation for the location of the critical line. It is argued that because the energy spectrum of *Casimir* vacuum will be governed by *Schrödinger* equation of quantum mechanics, in view of *Heisenberg* matrix mechanics physical space should be described by noncommutative spectral geometry of *Connes*. Invariant forms of transport coefficients suggesting finite values of gravitational viscosity as well as hierarchies of vacua and absolute zero temperatures are described. Some of the implications of the results to the problem of thermodynamic irreversibility and *Poincaré* recurrence theorem are addressed. Invariant modified form of the first law of thermodynamics is derived and a modified definition of entropy is introduced that closes the gap between radiation and gas theory. Finally, new paradigms for hydrodynamic foundations of both *Schrödinger* as well as *Dirac* wave equations and transitions between *Bohr* stationary states in quantum mechanics are discussed.**

Index Terms— **Kinetic theory of ideal gas; Thermodynamics; Statistical mechanics; Riemann hypothesis; TOE.**

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I. INTRODUCTION

It is well known that the methods of statistical mechanics can be applied to describe physical phenomena over a broad range of scales of space and time from the exceedingly large scale of cosmology to the minute scale of quantum optics as schematically shown in Fig. 1. All that is needed is that the system should contain a large number of weakly coupled particles. The similarities between stochastic quantum fields [1-17] and classical hydrodynamic fields [18-29] resulted in recent introduction of a scale-invariant model of statistical mechanics [30] and its application to thermodynamics [31, 32], fluid mechanics [33], and quantum mechanics [34].

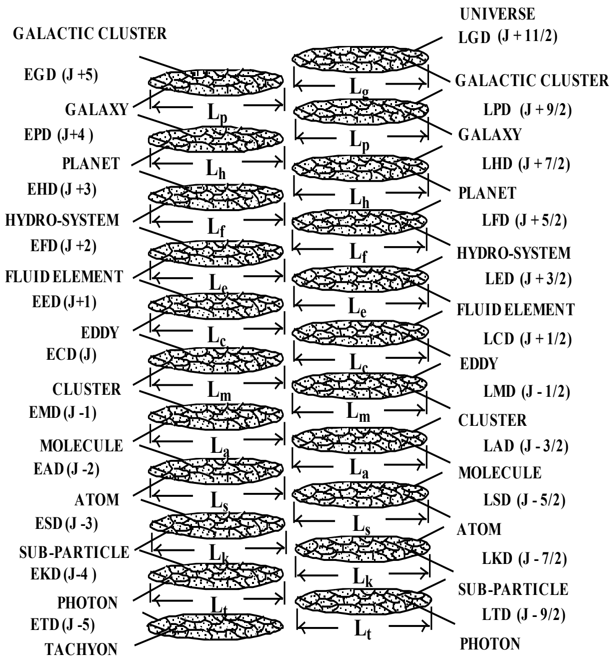


Fig. 1 Scale-invariant model of statistical mechanics. Equilibrium-β-Dynamics on the left-hand-side and non-equilibrium Laminar-β-Dynamics on the right-hand-side for scales β = g, p, h, f, e, c, m, a, s, k, and t as defined in Section 2. Characteristic lengths of (system, element, "atom") are (L_β, λ_β, ℓ_β) and λ_β is the mean-free-path [32].

In the present study [35] the invariant model of statistical mechanics and its implications to the physical foundations of thermodynamics, kinetic theory of ideal gas [36-42], and quantum mechanics are further examined. Whereas the outline of the main ideas are described in this introduction, references to most of the specific literature will be presented in the corresponding Sections.

After a brief introduction of a scale invariant model of statistical mechanics the invariant definitions of density, "atomic", element and system velocities are presented in Sec. 2. The invariant forms of conservation equations for mass, energy, linear and angular momentum based on linearization of *Boltzmann* equation and in harmony with *Enskog* [43] methods are described in Sec. 3. Because by definition fluids can only support normal stresses, following *Cauchy* a modified form of the stress tensor for fluids is introduced that leads to modified *Stokes* assumption thus a finite value of bulk viscosity such that in the limit of vanishing interatomic spacing all tangential stresses in the fluid vanish in accordance with the perceptions of *Cauchy* and *Poisson*. In addition, the concept of absolute enthalpy and iso-spin are introduced and incorporated in the derivation of scale invariant forms of energy and angular momentum conservation equations following the classical method of summational invariants [33]. The nature of the equation of motion for **(a)** equilibrium flow in absence of iso-spin **(b)** laminar potential flow **(c)** viscous flow in presence of spin are identified.

In Sec. 4 hierarchies of embedded statistical fields from *Planck* to cosmic scales are described. It is shown that the scale factor of 10^{-17} appears to separate the equilibrium statistical fields of chromodynamics (*Planck* scale), electrodynamics, hydrodynamics, planetary-dynamics (astrophysics), and galactic-dynamics (cosmology). The phenomenon of *Brownian* motion is described in terms of the statistical field of equilibrium cluster-dynamics ECD. The stochasticity of cascade of statistical fields is found to continue to *Casimir* vacuum that is identified as a tachyonic fluid that is *Dirac* stochastic ether or *de Broglie* hidden thermostat and is considered to be compressible in accordance with compressible ether of *Planck*. Stochastic definitions of *Planck* h and *Boltzmann* k constants are presented and shown to be respectively associated with the spatial and the temporal aspects of vacuum fluctuations and lead to finite gravitational mass of photon. Atomic mass unit is then identified as the total energy of photon thus suggesting that all baryonic matter is composed of light. It is shown that when thermodynamic temperature is modified by a factor of 1/2 based on the energy kT/2 per degree of freedom in accordance with *Boltzmann* equipartition principle, one resolves the classical *Newton* problem and obtains the velocity of sound in close agreement with observations [32]. The factor of 1/2 in the definition of temperature also results in modified *Joule-Mayer* mechanical equivalent of heat that is identified as the modified universal gas constant and is called *De Pretto* number 8338 (J/kcal) that appeared in the mass-energy equivalence equation of *De Pretto* [88].

In Sec. 5 invariant *Boltzmann* distribution function is derived by application of *Boltzmann* combinatoric method. The invariant *Planck* energy distribution is then derived directly from the invariant *Boltzmann* distribution in Sec. 6. The universality of invariant *Planck* energy distribution law from cosmic to photonic scales is described. Parallel to *Wien* displacement law for wavelength, a frequency displacement law is introduced and the connection between the speed of light and the root-mean-square speed of ideal photon gas is revealed. The important role of *Boltzmann* combinatoric method to the foundation of quantum mechanics is discussed. It is suggested that at a given temperature the *Maxwell-Boltzmann* distribution function could be viewed as spectrum of stochastically stationary sizes of particle clusters. Since according to the scale invariant model of statistical mechanics the "atom" of statistical field at scale β is identified as the most-probable cluster size of the lower scale $\beta-1$ (Fig. 13), the definitions of (electron, photon, neutrino) are introduced as the most-probable equilibrium sizes of (photon, neutrino, tachyon) clusters. Also, definitions of both dark energy (electromagnetic mass) and dark matter (gravitational mass) are introduced.

Next *Maxwell-Boltzmann* speed distribution is directly derived from invariant *Planck* energy distribution in Sec. 7. Hence, at thermodynamic equilibrium particles of statistical fields of all scales (Fig. 1) will have *Gaussian* velocity distribution, *Planck* energy distribution, and *Maxwell-Boltzmann* speed distribution. *Montgomery-Odlyzko* law of correspondence between distribution of normalized spacing of zeros of *Riemann* zeta function and those of eigenvalues of *Gaussian* unitary ensemble (GUE) is shown to extend to normalized *Maxwell-Boltzmann* distribution function in Sec. 8. Thus, a connection is established between analytic number theory on the one hand and the kinetic theory of ideal gas on the other hand. The spacing between energy levels are then related to frequency spacing through *Planck* formula for quantum of energy $\varepsilon = h\nu$. Next, the frequencies of *Heisenberg-Kramers* virtual oscillators are taken as powers of prime numbers and expressed in terms of *Gauss's* clock calculators or *Hensel's* *p*-adic numbers. Finally, the spacing between zeros of particle velocities are related to zeros of *Riemann* zeta function through *Euler's* golden key. In addition, it is argued that since physical space or *Casimir* vacuum is identified as a tachyonic quantum fluid with energy spectra given by *Schrödinger* equation and hence *Heisenberg* matrix mechanics, it should be described by noncommutative spectral geometry of *Connes*.

The implication of invariant model of statistical mechanics to transport phenomena is addressed in Sec. 9. Following *Maxwell*, invariant definition of kinematic viscosity is presented that gives *Boussinesq* eddy viscosity for isotropic turbulence at the scale of equilibrium eddy-dynamics. The scale invariance of the model suggests possible dissipative effects at the much smaller scales of electrodynamics and chromodynamics. Hierarchies of "absolute zero" thermodynamic temperatures and associated vacua in harmony with inflationary models of cosmology are described. Also, the impact of *Poincaré* [44] recurrence theory on the problem of irreversibility in thermodynamics is discussed.

The derivation of invariant form of the first law of thermodynamics and modified definition of entropy per photon are presented in Sec. 10. It is shown that space quantization leads to a modified expression for the number of photons in a given volume resulting in exact correspondence between photon gas and the classical monatomic ideal gas thus closing the gap between radiation and gas theory.

The derivation of invariant *Schrödinger* equation from invariant *Bernoulli* equation for potential incompressible flow is discussed in Sec. 11. A new paradigm of physical foundation of quantum mechanics is presented according to which *Bohr* stationary states correspond to statistically stationary sizes of particle clusters, *de Broglie* wave packets, which are governed by *Maxwell-Boltzmann* distribution function. Finally, invariant *Dirac* relativistic wave equation and its derivation from invariant equation of motion in the presence of viscous effects hence iso-spin is described.

II. A Scale Invariant Model of Statistical Mechanics

The scale-invariant model of statistical mechanics for equilibrium galactic-, planetary-, hydro-system-, fluid-element-, eddy-, cluster-, molecular-, atomic-, subatomic-, kromo-, and tachyon-dynamics corresponding to the scale β = g, p, h, f, e, c, m, a, s, k, and t is schematically shown in Fig. 1 [32]. Each statistical field is identified as the "system" and is composed of a spectrum of "elements". Each element is composed of an ensemble of small particles called the "atoms" of the field that are governed by distribution function $f_{i\beta}(\mathbf{x}_{i\beta}, \mathbf{u}_{i\beta}, t_\beta)$ and viewed as *point-mass*. The most probable element (system) velocity of the smaller scale *j* becomes the velocity of the atom (element) of the larger scale *j+1* [34]. Since invariant Schrödinger equation was recently derived from invariant Bernoulli equation [34], the entire hierarchy of statistical fields shown in Fig. 1 is governed by quantum mechanics. There are no physical or mathematical reasons for the hierarchy shown in Fig. 1 not to continue to larger and smaller scales ad infinitum. Hence, according to Fig. 1 contrary to the often quoted statement by Einstein that God does not play dice; the Almighty appears to be playing with infinite hierarchies of embedded dice.

Following the classical methods [43, 45-49] the invariant definitions of density ρ_β and velocity of "*atom*" $\mathbf{u}_\beta$, *element* $\mathbf{v}_\beta$, and *system* $\mathbf{w}_\beta$ at the scale β are given as [33]

$$\rho_\beta = n_\beta m_\beta = m_\beta \int f_\beta du_\beta \quad , \quad \mathbf{u}_\beta = \mathbf{v}_{mp\beta-1} \tag{1}$$

$$\mathbf{v}_\beta = \rho_\beta^{-1} m_\beta \int \mathbf{u}_\beta f_\beta d\mathbf{u}_\beta \quad , \quad \mathbf{w}_\beta = \mathbf{v}_{mp\beta+1} \tag{2}$$

Similarly, the invariant definition of the peculiar and diffusion velocities are introduced as

$$\mathbf{V}'_\beta = \mathbf{u}_\beta - \mathbf{v}_\beta \quad , \quad \mathbf{V}_\beta = \mathbf{v}_\beta - \mathbf{w}_\beta \tag{3}$$

such that

$$\mathbf{V}_\beta = \mathbf{V}'_{\beta+1} \tag{4}$$

For each statistical field, one defines particles that form the background fluid and are viewed as point-mass or "*atom*" of the field. Next, the *elements* of the field are defined as finite-sized composite entities each composed of an ensemble of "*atoms*" as shown in Fig. 1. According to equations (1)-(2) the atomic and system velocities of scale β $(\mathbf{u}_\beta, \mathbf{w}_\beta)$ are the most-probable speeds of the lower and upper adjacent scales $(\mathbf{v}_{mp\beta-1}, \mathbf{v}_{mp\beta+1})$ as shown in Fig. 13. Finally, the ensemble of a large number of "*elements*" is defined as the statistical "*system*" at that particular scale.

III. Scale Invariant Forms of Conservation Equations

Following the classical methods [43, 45-49] the scale-invariant forms of mass, thermal energy, linear and angular momentum conservation equations at scale β are given as [33, 50]

$$\frac{\partial \rho_{i\beta}}{\partial t_\beta} + \nabla.(\rho_{i\beta}\mathbf{v}_{i\beta}) = \Re_{i\beta} \tag{5}$$

$$\frac{\partial \varepsilon_{i\beta}}{\partial t_\beta} + \nabla.(\varepsilon_{i\beta}\mathbf{v}_{i\beta}) = 0 \tag{6}$$

$$\frac{\partial \mathbf{p}_{i\beta}}{\partial t_\beta} + \nabla.(\mathbf{p}_{i\beta}\mathbf{v}_{j\beta}) = -\nabla.\mathbf{P}_{ij\beta} \tag{7}$$

$$\frac{\partial \boldsymbol{\pi}_{i\beta}}{\partial t_\beta} + \nabla.(\boldsymbol{\pi}_{i\beta}\mathbf{v}_{i\beta}) = \rho_{i\beta}\boldsymbol{\omega}_\beta.\nabla\mathbf{v}_{i\beta} \tag{8}$$

that involve the *volumetric density* of thermal energy $\varepsilon_{i\beta} = \rho_{i\beta}\bar{h}_{i\beta} = \tilde{\rho}_{i\beta}\tilde{h}_{i\beta}$, linear momentum $\boldsymbol{p}_{i\beta} = \rho_{i\beta}\mathbf{v}_{i\beta}$, $\boldsymbol{\pi}_{i\beta} = \rho_{i\beta}\boldsymbol{\omega}_{i\beta}$, and vorticity $\boldsymbol{\omega}_{i\beta} = \nabla\times\mathbf{v}_{i\beta}$. Also, $\Re_{i\beta}$ is the chemical reaction rate, $\tilde{h}_{i\beta}$ is the absolute enthalpy [33]

$$\tilde{h}_{i\beta} = \int_0^T \tilde{c}_{pi\beta} dT_\beta \tag{9}$$

and $\mathbf{P}_{ij\beta}$ is the stress tensor [45]

$$\mathbf{P}_{ij\beta} = m_\beta \int (\mathbf{u}_{i\beta} - \mathbf{v}_{i\beta})(\mathbf{u}_{j\beta} - \mathbf{v}_{j\beta}) f_\beta du_\beta \tag{10}$$

Derivation of Eq. (7) is based on the definition of the peculiar velocity in Eq. (3) along with the identity

$$\overline{\mathbf{V}'_{i\beta}\mathbf{V}'_{j\beta}} = \overline{(\mathbf{u}_{i\beta} - \mathbf{v}_{i\beta i})(\mathbf{u}_{j\beta} - \mathbf{v}_{j\beta})} = \overline{\mathbf{u}_{i\beta}\mathbf{u}_{j\beta}} - \mathbf{v}_{i\beta}\mathbf{v}_{j\beta} \tag{11}$$

The definition of absolute enthalpy in Eq. (9) results in the definition of standard heat of formation $\tilde{h}^o_{if\beta}$ for chemical specie i [33]

$$\tilde{h}^o_{fi\beta} = \int_0^{T_o} \tilde{c}_{pi\beta} dT_\beta \tag{12}$$

where T_o is the standard temperature. The definition in Eq. (12) helps to avoid the conventional practice of arbitrarily setting the standard heat of formation of naturally occurring species equal to zero. Furthermore, following *Nernst-Planck* statement of the third law of thermodynamics one has $\tilde{h}_\beta \to 0$ in the limit $T_\beta \to 0$ as expected.

The classical definition of vorticity involves the curl of linear velocity $\nabla\times\mathbf{v}_\beta = \boldsymbol{\omega}_\beta$ thus giving rotational velocity of particle a secondary status in that it depends on its translational velocity $\mathbf{v}_\beta$. However, it is known that particle's rotation about its center of mass is independent of the translational motion of its center of mass. In other words, translational, rotational, and vibrational (pulsational) motions of particle are independent degrees of freedom that should not be necessarily coupled. To resolve this paradox, the iso-spin of particle at scale β is defined as the curl of the velocity at the next lower scale of β–1 [51]

$$\boldsymbol{\varpi}_\beta = \nabla \times \mathbf{v}_{\beta-1} = \boldsymbol{\omega}_{\beta-1} = \nabla \times \mathbf{u}_\beta \tag{13}$$

such that the rotational velocity, while having a connection to some type of translational motion at internal scale β–1, retains its independent degree of freedom at the external scale β as desired. A schematic description of iso-spin and vorticity fields is shown in Fig. 2. The nature of galactic vortices in cosmology and the associated dissipation have been discussed [25, 52].

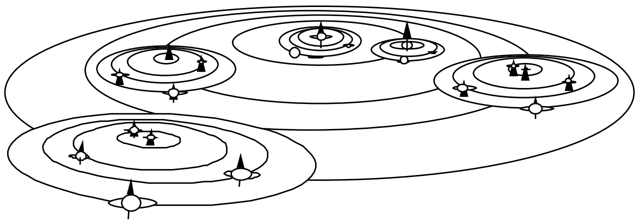


Fig. 2 Description of internal (iso-spin) versus external vorticity fields in cosmology [51].

The local velocity $\mathbf{v}_\beta$ in equations (5)-(8) is expressed in terms of the convective $\mathbf{w}_\beta$ and the diffusive $\mathbf{V}_\beta$ velocities [33]

$$\mathbf{w}_\beta = \mathbf{v}_\beta - \mathbf{V}_{\beta g} \quad , \quad \mathbf{V}_{\beta g} = -D_\beta \nabla \ln(\rho_\beta) \tag{14a}$$

$$\mathbf{w}_\beta = \mathbf{v}_\beta - \mathbf{V}_{\beta tg} \quad , \quad \mathbf{V}_{\beta tg} = -\alpha_\beta \nabla \ln(\varepsilon_\beta) \tag{14b}$$

$$\mathbf{w}_\beta = \mathbf{v}_\beta - \mathbf{V}_{\beta hg} \quad , \quad \mathbf{V}_{\beta hg} = -\kappa_\beta \nabla \ln(\mathbf{p}_\beta) \tag{14c}$$

$$\mathbf{w}_\beta = \mathbf{v}_\beta - \mathbf{V}_{\beta rhg} \quad , \quad \mathbf{V}_{\beta rhg} = -\kappa_\beta \nabla \ln(\boldsymbol{\pi}_\beta) \tag{14d}$$

where D_β is mass diffusivity, $\kappa_\beta = \eta_\beta / \rho_\beta$ is kinematic viscosity, η_β is dynamic viscosity, $\alpha_{i\beta} = k_\beta / \rho_\beta c_p$ and k_β are thermal diffusivity and conductivity, and $(\mathbf{V}_{\beta g}, \mathbf{V}_{\beta tg}, \mathbf{V}_{\beta hg}, \mathbf{V}_{\beta rhg})$ are respectively the diffusive, the thermo-diffusive, the translational and rotational hydro-diffusive velocities.

Because by definition fluids can only support compressive normal forces, following *Cauchy* the stress tensor for fluids is first expressed as [33]

$$\mathbf{P}_{ij\beta} = \frac{1}{3}\sigma_{ii\beta}\delta_{ij\beta} = p_\beta \delta_{ij\beta} + (\lambda'_{i\beta} + \frac{2}{3}\eta_{i\beta})\,\nabla.\mathbf{v}_{i\beta}\delta_{ij} \tag{15}$$

and the classical *Stokes* assumption [53] of zero bulk viscosity $b_\beta = 0$ is modified such that the two *Lame* constants $(\lambda'_\beta, \ \eta_\beta)$ lead to finite bulk viscosity [33]

$$b_\beta = \lambda'_\beta + \frac{2}{3}\eta_\beta = -\frac{\eta_\beta}{3} \tag{16}$$

that by Eq. (15) result in the total stress tensor [33]

$$\mathbf{P}_{ij\beta} = p_\beta \delta_{ij\beta} - \frac{1}{3}\eta_\beta \nabla.\mathbf{v}_\beta \delta_{ij\beta} = (p_{t\beta} + p_{h\beta})\delta_{ij\beta} \tag{17}$$

involving thermodynamic $p_{t\beta}$ and hydrodynamic $p_{h\beta}$ pressures.

The expression for hydrodynamic pressure in Eq. (17) could also be arrived at directly by first noting that classically hydrodynamic pressure is defined as the mean normal stress

$$p_{h\beta} = (\tau_{xx\beta} + \tau_{yy\beta} + \tau_{zz\beta}) / 3 \tag{18}$$

since shear stresses in fluids vanish by definition. Next, normal stresses are expressed as diffusional flux of the corresponding momenta by Eq. (14c) as

$$\tau_{ii\beta} = \rho_{i\beta}\mathbf{v}_{i\beta}\mathbf{V}_{i\beta h} = -\eta_\beta \nabla.\mathbf{v}_{i\beta} \tag{19}$$

Substituting from Eq. (19) into Eq. (18) results in

$$p_{h\beta} = \frac{1}{3}(\tau_{xx\beta} + \tau_{yy\beta} + \tau_{zz\beta}) = -\frac{1}{3}\eta_\beta \nabla.\mathbf{v}_\beta \tag{20}$$

that is in accordance with Eq. (17). The occurrence of a single rather than two *Lame* constants in Eq. (16) is in accordance with the perceptions of *Cauchy* and *Poisson* who both assumed the limit of zero for the expression [54]

$$\lambda'_\beta + \eta_\beta = \lim_{R \to 0} R^4 f(R) \tag{21}$$

It is because of Eq. (21) that the *Stokes* assumption in Eq. (16) is equal to $-\eta_\beta / 3$ rather than zero and as the intermolecular spacing vanishes $R \to 0$ by Eq. (21) all stresses become normal in accordance with Eqs. (17)-(20). Because in the limit given in equation (21) as was noted by *Darrigol* [54]

"Then the medium loses its rigidity since the transverse pressures disappear."

one may identify the medium in the limit in Eq. (21) as fluid requiring only a single *Lame* coefficient as anticipated by *Navier* [54].

Following the classical methods [43, 45-47] by substituting from equations (14)-(17) into (5)-(8) and neglecting cross-diffusion terms and assuming constant transport coefficients the invariant forms of conservation equations are written as [33]

$$\frac{\partial \rho_{i\beta}}{\partial t_\beta} + \mathbf{w}_\beta.\nabla \rho_{i\beta} = D_{i\beta}\nabla^2 \rho_{i\beta} + \Re_{i\beta} \tag{22}$$

$$\frac{\partial T_{i\beta}}{\partial t_\beta} + \mathbf{w}_\beta.\nabla T_{i\beta} = \alpha_\beta \nabla^2 T_{i\beta} - \tilde{h}_{i\beta}\Re_{i\beta} / (\rho_{i\beta} c_{pi\beta}) \tag{23}$$

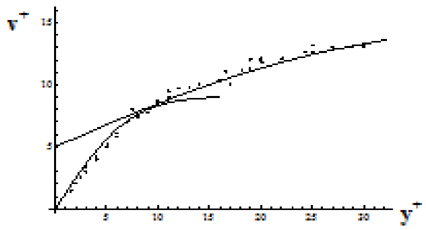
$$\frac{\partial \mathbf{v}_{i\beta}}{\partial t_\beta} + \mathbf{w}_\beta . \nabla \mathbf{v}_{i\beta} = \kappa_{i\beta} \nabla^2 \mathbf{v}_{i\beta} - \frac{\nabla p_{i\beta}}{\rho_{i\beta}} + \frac{1}{3} \kappa_{i\beta} \nabla(\nabla . \mathbf{v}_{i\beta}) - \frac{\mathbf{v}_{i\beta} \Re_{i\beta}}{\rho_{i\beta}} \quad (24)$$

$$\frac{\partial \boldsymbol{\omega}_{i\beta}}{\partial t_\beta} + \mathbf{w}_\beta . \nabla \boldsymbol{\omega}_{i\beta} = \kappa_{i\beta} \nabla^2 \boldsymbol{\omega}_{i\beta} + \boldsymbol{\omega}_\beta . \nabla \mathbf{v}_{i\beta} - \frac{\boldsymbol{\omega}_{i\beta} \Re_{i\beta}}{\rho_{i\beta}} \quad (25)$$

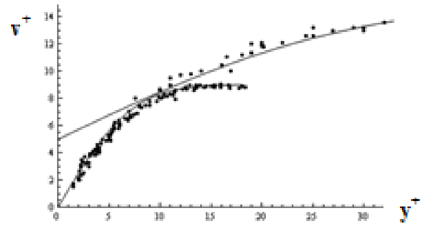
The modified form of the equation of motion in (24) is to be compared with the Navier-Stokes equation of motion

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}.\nabla \mathbf{v} = \kappa \nabla^2 \mathbf{v} - \frac{\nabla p}{\rho} + \frac{1}{3} \kappa \nabla(\nabla . \mathbf{v}) \quad (26)$$

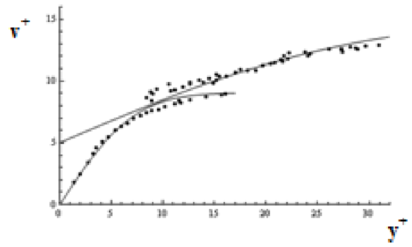
An important difference between equations (24) and (26) is the occurrence of the convective velocity $\mathbf{w}_\beta$ versus the local velocity $\mathbf{v}_\beta$ in the second term of the former similar to Carrier equation [33, 56]. One notes that in the absence of convection Eq. (24) reduces to nonhomogeneous diffusion equation similar to Eqs. (22) and (23). However, the absence of convention results in vanishing of almost the entire classical equation of motion (26). Similarly, an important difference between the modified (25) and the classical forms of *Helmholtz* vorticity equation is the occurrence of convective velocity $\mathbf{w}_\beta$ as opposed to local velocity $\mathbf{v}_\beta$ in the second term of Eq. (25) [33, 56]. Because local vorticity $\boldsymbol{\omega}_\beta$ in (25) is itself related to the curl of local velocity it cannot be convected by this same velocity. On the other hand, the advection of local vorticity by convective velocity $\mathbf{w}_\beta$ is possible. Moreover, in absence of convection Eq. (25) reduces to the diffusion equation similar to that in (21)-(24) for mass, heat, and momentum, transport [33, 56]. The solution of Eq. (24) for the problem of turbulent flow over a flat plate [56, 57] at LED, LCD, LMD, and LAD scales are given in Figs. 3a-3c.



(a)



(b)



(c)

Fig. 3 Comparison between the predicted velocity profiles (a) LED-LCD, (b) LCD-LMD, (c) LMD-LAD with experimental data in the literature over 10^8 range of spatial scales [34].

The central question concerning Cauchy equation of motion Eq. (7) say at β = m is how many "molecules" are included in the definition of the mean molecular velocity $\mathbf{v}_{mj} = <\mathbf{u}_{mj}> = \mathbf{u}_{cj}$. One can identify three distinguishable cases:

(a) When $\mathbf{v}_m = \mathbf{u}_c$ is itself random then all three velocities $(\mathbf{u}_m, \mathbf{v}_m, \mathbf{V}'_m)$ in Eq. (3) are random in a stochastically stationary field made of ensembles of clusters and molecules with *Brownian* motions and hence *Gaussian* velocity distribution, *Planck* energy distribution, and *Maxwell-Boltzmann* speed distribution. If in addition both the vorticity $\boldsymbol{\omega}_m = \nabla \times \mathbf{v}_m = 0$ as well as the iso-spin $\boldsymbol{\varpi}_m = \nabla \times \mathbf{u}_m = 0$ are zero then for an incompressible flow the continuity Eq. (5) and *Cauchy* equation of motion (7) lead to *Bernoulli* equation. In Sec. 11 it will be shown that under the above mentioned conditions *Schrödinger* equation (206) can be directly derived [34] from *Bernoulli* equation (202) such that the energy spectrum of the equilibrium field will be governed by quantum mechanics and hence by *Planck* law.

(b) When $\mathbf{v}_m = \mathbf{u}_c$ is not random but the vorticity vanishes $\boldsymbol{\omega}_m = 0$ the flow is irrotational and ideal, inviscid $\eta_m = 0$, and once again one obtains *Bernoulli* equation from equations (5) and (7) with the solution given by the classical potential flow.

(c) When $\mathbf{v}_m = \mathbf{u}_c$ is not random and vorticity does not vanish $\nabla \times \mathbf{v}_m \neq 0$ the rotational non-ideal viscous $\eta_m \neq 0$ flow will be governed by the equation of motion (24) with the convection velocity $\mathbf{w}_\beta = \mathbf{v}_{\beta+1}$ obtained from the solution of potential flow at the next larger scale of β+1. In Sec. 11 it will be shown that the viscous equation of motion Eq. (24) is associated with *Dirac* relativistic wave equation. In the sequel, amongst the three cases of flow conditions discussed in Ref. 34 only cases **(a)** and **(b)** will be examined.

IV. Hierarchies of Embedded Statistical Fields

The invariant model of statistical mechanics shown in Fig. 1 and described by equations (1)-(4) suggests that all statistical fields are turbulent fields and governed by equations (5)-(8) [33, 34]. First, let us start with the field of laminar molecular dynamics LMD when molecules, clusters of molecules (cluster), and cluster of clusters of molecules (eddy) form the "atom", the "element", and the "system" with the velocities $(\mathbf{u}_m, \mathbf{v}_m, \mathbf{w}_m)$. Similarly, the fields of laminar cluster-dynamics LCD and eddy-dynamics LED will have the velocities $(\mathbf{u}_c, \mathbf{v}_c, \mathbf{w}_c)$ and $(\mathbf{u}_e, \mathbf{v}_e, \mathbf{w}_e)$ in accordance with equations (1)-(2). For the fields of EED, ECD, and EMD typical characteristic "atom", element, and system lengths are [50]

$$\text{EED} \quad (\ell_e, \lambda_e, L_e) = (10^{-5}, 10^{-3}, 10^{-1}) \text{ m} \qquad (27a)$$

$$\text{ECD} \quad (\ell_c, \lambda_c, L_c) = (10^{-7}, 10^{-5}, 10^{-3}) \text{ m} \qquad (27b)$$

$$\text{EMD} \quad (\ell_m, \lambda_m, L_m) = (10^{-9}, 10^{-7}, 10^{-5}) \text{ m} \qquad (27c)$$

If one applies the same (atom, element, system) = $(\ell_\beta, \lambda_\beta, L_\beta)$ relative sizes in Eq. (27) to the entire spatial scale of Fig. 1 and considers the relation between scales as $\ell_\beta = \lambda_{\beta-1} = L_{\beta-2}$ then the resulting cascades or hierarchy of overlapping statistical fields will appear as schematically shown in Fig. 4. According to Fig. 4, starting from the hydrodynamic scale $(10^3, 10^1, 10^{-1}, 10^{-3})$ after seven generations of statistical fields one reaches the electro-dynamic scale with the element size 10^{-17} and exactly after seven more generations one reaches *Planck* length scale $(\hbar G / c^3)^{1/2} \approx 10^{-35}\,\text{m}$, where G is the gravitational constant. Similarly, seven generations of statistical fields separate the hydrodynamic scale $(10^3, 10^1, 10^{-1}, 10^{-3})$ from the scale of planetary dynamics (astrophysics) 10^{17} and the latter from galactic-dynamics (cosmology) 10^{35} m. There are no physical or mathematical reasons for the hierarchy shown in Fig. 4 not to continue to larger and smaller scales ad infinitum. Hence, according to Fig. 4 contrary to the often quoted statement by *Einstein* that God does not play dice; the Almighty appears to be playing with infinite hierarchies of embedded dices.

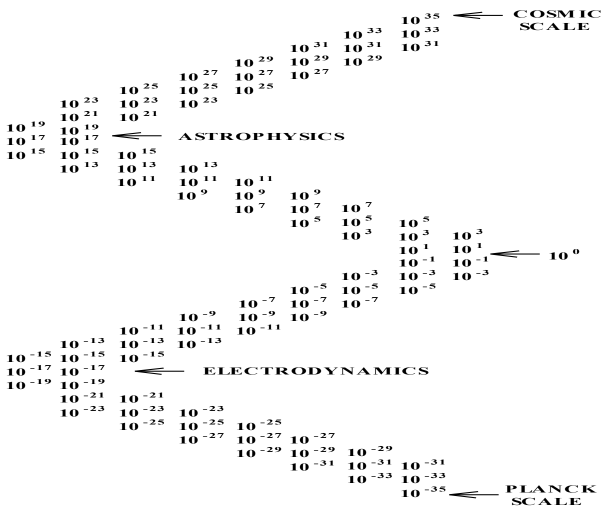


Fig. 4 Hierarchy of statistical fields with $(\ell_\beta, \lambda_\beta, L_\beta)$ **from cosmic to Planck scales [34].**

The left hand side of Fig. 1 corresponds to equilibrium statistical fields when the velocities of elements of the field are random since at thermodynamic equilibrium particles i.e. oscillators of such statistical fields will have normal or *Gaussian* velocity distribution. For example, for stationary homogeneous isotropic turbulence at EED scale the experimental data of *Townsend* [58] confirms *Gaussian* velocity distribution of eddies as shown in Fig. 5.

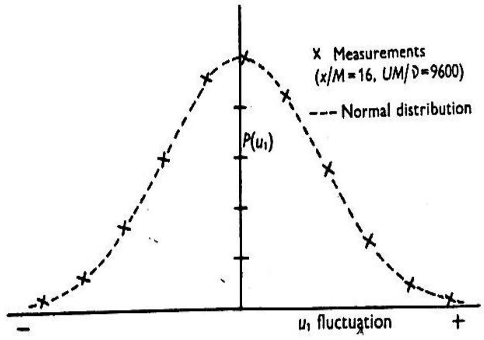


Fig. 5 Measured velocity distribution in isotropic turbulent flow [58].

According to Fig. 1, the statistical fields of equilibrium eddy-dynamics and molecular-dynamics are separated by the equilibrium field of cluster-dynamics at an intermediate scale. The evidence for the existence of the statistical field of equilibrium cluster-dynamics ECD (Fig. 1) is the phenomena of *Brownian* motions [25, 59-69]. Modern theory of *Brownian* motion starts with *Langevin* equation [25]

$$\frac{d\mathbf{u}_p}{dt} = -\bar{\beta}\mathbf{u}_p + \mathbf{A}(t) \qquad (28)$$

where $\mathbf{u}_p$ is the particle velocity. The drastic nature of the assumptions inherent in the division of forces in Eq. (28) was emphasized by *Chandrasekhar* [25].

To account for the stationary nature of *Brownian* motions fluid fluctuations at scales much larger than molecular scales are needed as noted by *Gouy* [59]. Observations have shown that as the size of the particles decrease their movement become faster [59]. According to classical arguments *Brownian* motions are induced by multiple collisions of a large number of molecules with individual suspended particle. However, since the typical size of particle is about 100 times larger than that of individual molecules, such collisions preferentially from one side of the particle could not occur in view of the assumed *Maxwell-Boltzmann* distribution of molecular motions. On the other hand, if one assumes that *Brownian* motions are induced by collisions of particles with groups, i.e. clusters, of molecules then in view of the stationary nature of *Brownian* motions, the motions of such clusters themselves must also be governed by *Maxwell-Boltzmann* distribution. But this would mean the existence of the statistical field of equilibrium cluster dynamics.

The description of *Brownian* motions as equilibrium between suspended particles and a spectrum of molecular clusters that themselves possess *Brownian* motions resolves the paradox associated with the absence of dissipation and hence violation of the *Carnot* principle or *Maxwell's* demon paradox emphasized by *Poincaré* [70]

"M. Gouy had the idea of looking a little more closely, and thought he saw that this explanation was untenable; that the motion becomes more active as the particle become smaller, but that they are uninfluenced by the manner of lighting. If, then, these motions do not cease, or, rather, if they come into existence incessantly, without borrowing from any external source of energy, what must we think? We must surely not abandon on this account the conservation of energy; but we see before our eyes motion transformed into heat by friction and conversely heat changing into motion, and all without any sort of loss, since the motion continues forever. It is the contradiction of Carnot's principle. If such is the case, we need no longer the infinitely keen eye of Maxwell's demon in order to see the world move backward; our microscope suffices."

Therefore, as was anticipated by *Poincaré* [70] the revolution caused by violation of the second law of thermodynamics due to *Brownian* motions namely *Maxwell's* demon paradox is just as great as that due to his Principle of Relativity [70]. The dynamic theory of relativity of *Poincaré -Lorentz* that is *causal* since it is induced by compressibility of tachyonic fluid that constitutes the physical space (ether) as opposed to the kinematic theory of relativity of *Einstein* were described in a recent study [34].

Clearly, the concept of spectrum of molecular clusters undergoing *Brownian* motions in ECD (Fig. 1) is in harmony with the perceptions of *Sutherland* [62] regarding "ionic aggregates" or large "molecular masses". The central importance of the work of *Sutherland* [62] on *Brownian* motion and its impact on the subsequent work of *Einstein* is evidenced by the correspondence between *Einstein* and his friend *Michele Besso* in 1903 described in the excellent study on history and modern developments of the theories of *Brownian* motion by *Duplantier* [71]

"In 1903, Einstein and his friend Michele Besso discussed a theory of dissociation that required the assumption of molecular aggregates in combination with water, the "hypothesis of ionic aggregates", as Besso called it. This assumption opens the way to a simple calculation of the sizes of ions in solution, based on hydrodynamical considerations. In 1902, Sutherland had considered in Ionization, Ionic Velocities, and Atomic Sizes[38] a calculation of the sizes of ions on the basis of Stokes' law, but criticized it as in disagreement with experimental data[39]. The very same idea of determining sizes of ions by means of classical hydrodynamics occurred to Einstein in his letter of 17 March 1903 to Besso[40], where he reported what appears to be just the calculation that Sutherland had performed,"

The importance of *Sutherland's* earlier 1902 work [62] on ionic sizes have also been emphasized [71]

"However, upon reading these letters of 1903, one cannot refrain from wondering whether Besso and Einstein were not also acquainted with and discussing Sutherland's 1902 paper on ionic sizes. In that case, Sutherland suggestion to use hydrodynamic Stokes' law to determine the size of molecules would have been a direct inspiration to Einstein's dissertation and subsequent work on Brownian motion!"

Because of *Sutherland*'s pioneering contributions to the understanding of *Brownian* motion the dual name *Sutherland-Einstein* is to be associated with the expression for diffusion coefficient [62, 64]

$$D = \frac{R^o T}{N^o}\frac{1}{6\pi\eta r_a} \tag{29a}$$

as discussed by *Duplantier* [71]

"In this year 2005, it is definitely time, I think, for the physics community to finally recognize Sutherland's achievements, and following Pais' suggestion, to re-baptize the famous relation (2) with a double name!"

In equation (29a) r_a is "atomic" radius, R^o is the universal gas constant, and N^o is the *Avogadro-Loschmidt* number. According to the ultra-simplified model of ideal gas [49] the diffusion coefficient becomes identical to the molecular kinematic viscosity $\kappa = \eta/\rho = D$ given by *Maxwell* relation $\kappa_\beta = \ell_{\beta x} u_{\beta x+} / 3$ from Eq. (146) of Sec. 9. It is therefore interesting to examine if the equality $D = \kappa$ is also satisfied by *Sutherland-Einstein* relation in Eq. (29a). Although equation (29a) at first appears to suggest that D and κ are inversely related, the equality of mass and momentum diffusivity becomes evident when Eq. (29a) is expressed as

$$D = \frac{\rho RT}{N^o}\frac{1}{6\pi\eta r_a \rho} = \frac{p}{6\pi\eta r_a \rho N^o} = \frac{\eta}{\rho}\frac{p}{6\pi\eta^2 r_a N^o}$$

$$= \frac{\eta}{\rho}\frac{p}{6\pi\eta^2 r_a N^o} = \frac{\kappa p}{6\pi\eta^2 r_a N^o} = \frac{\kappa p}{6\pi\kappa^2\rho^2 r_a N^o}$$

$$= \frac{\kappa p}{6\pi(\frac{1}{3}\ell_{mx}u_{mx+})^2\rho^2 r_a N^o} = \frac{\kappa p}{\frac{\pi}{3}\ell_{mx}^2 r_a \rho N^o(\rho u_{mx}^2)}$$

$$= \frac{\kappa p}{\frac{\pi}{3}\lambda_{ax}^2 r_a \rho N^o(\frac{1}{3}\rho u_m^2)} = \frac{\kappa p}{\frac{\pi}{3}\lambda_{ax}^2 r_a \rho N^o p}$$

$$= \frac{\kappa}{\frac{4\pi}{3}r_a^3\rho N^o} = \frac{\kappa}{\rho\hat{v}N^o} = \frac{\kappa}{(amu)N^o} = \kappa \tag{29b}$$

In Eq. (29b) substitutions have been made for the kinematic viscosity $\kappa_m = \ell_{mx} u_{mx+} / 3$, the *Avogadro-Loschmidt* number N^o from Eq. (38), and $u_{mx}^2 = 2u_{mx+}^2$. Also, when the atomic mean free path is taken as atomic diameter $\ell_{mx} = \lambda_{ax} = 2r_a$, atomic volume $\hat{v} = 4\pi r_a^3 / 3$ results in $\rho\hat{v} =$ amu that is the atomic mass unit defined in Eq. (40).

Because at thermodynamic equilibrium the mean velocity of each particle or *Heisenberg-Kramers* [72] virtual oscillator vanishes $<\mathbf{u}_\beta> = 0$, the translational kinetic energy of particle oscillating in two directions (x+, x–) is expressed as

$$\varepsilon_\beta = m_\beta \langle u_{\beta x+}^2 \rangle / 2 + m_\beta \langle u_{\beta x-}^2 \rangle / 2 = m_\beta \langle u_{\beta x+}^2 \rangle = \langle p'_\beta \rangle \langle \lambda_\beta^2 \rangle^{1/2} \langle \nu_\beta^2 \rangle^{1/2} \quad (30)$$

where $< p'_\beta > = m_\beta < u_{\beta x+}^2 >^{1/2}$ is the root-mean-square momentum of particle and $<u^2_{\beta x+}> = <u^2_{\beta x-}>$ by *Boltzmann* equipartition principle. At any scale β the result in Eq. (30) can be expressed in terms of either frequency or wavelength

$$\varepsilon_\beta = m_\beta \langle u_\beta^2 \rangle = \langle p'_\beta \rangle \langle \lambda_\beta^2 \rangle^{1/2} \langle \nu_\beta^2 \rangle^{1/2} = h_\beta \nu_\beta \quad (31a)$$

$$\varepsilon_\beta = m_\beta \langle u_\beta^2 \rangle = \langle p'_\beta \rangle \langle \nu_\beta^2 \rangle^{1/2} \langle \lambda_\beta^2 \rangle^{1/2} = k_\beta \lambda_\beta \quad (31b)$$

when the definition of stochastic *Planck* and *Boltzmann* factors are introduced as [34]

$$h_\beta = \langle p'_\beta \rangle \langle \lambda_\beta^2 \rangle^{1/2} \quad (32a)$$

$$k_\beta = \langle p'_\beta \rangle \langle \nu_\beta^2 \rangle^{1/2} \quad (32b)$$

At the important scale of EKD (Fig. 1) corresponding to *Casimir* [73] vacuum composed of photon gas, the universal constants of *Planck* [74, 75] and *Boltzmann* [31] are identified from equations (31)-(32) as

$$h = h_k = m_k c \langle \lambda_k^2 \rangle^{1/2} = 6.626 \times 10^{-34} \text{ J/s} \quad (33a)$$

$$k = k_k = m_k c \langle \nu_k^2 \rangle^{1/2} = 1.381 \times 10^{-23} \text{ J/K} \quad (33b)$$

Next, following *de Broglie* hypothesis for the wavelength of matter waves [2]

$$\lambda_\beta = h / p'_\beta \quad (34)$$

the frequency of matter waves is defined as [31]

$$\nu_\beta = k / p'_\beta \quad (35)$$

For matter and radiation in the state of thermodynamic equilibrium equations (32) and (33) can be expressed as

$$h_\beta = h_k = h \quad , \quad k_\beta = k_k = k \quad (36)$$

The definitions in equations (34)-(35) result in the gravitational mass of photon [31]

$$m_k = (hk / c^3)^{1/2} = 1.84278 \times 10^{-41} \text{ kg} \quad (37)$$

that is much larger than the reported [76] value of 4×10^{-51} kg. The finite gravitational mass of photons was anticipated by *Newton* [77] and is in accordance with *Einstein-de Broglie* [78-82] theory of light. *Avogardo-Loschmidt* number was predicted as [31]

$$N^o = 1/(m_k c^2) = 6.0376 \times 10^{23} \quad (38)$$

leading to the modified value of the universal gas constant

$$R^o = N^o k = 8.338 \text{ J/(kmol-k)} \quad (39)$$

Also, the atomic mass unit is obtained from equations (37)-(38) as

$$\text{amu} = m_k c^2 = (hkc)^{1/2} = 1.6563 \times 10^{-27} \text{ kg/kmol} \quad (40)$$

Since all baryonic matter is known to be composed of atoms, the results in equations (38) and (40) suggest that all matter in the universe is composed of light [83]. From equations (32)-(33) the wavelength and frequency of photon in vacuum $\langle \lambda_k^2 \rangle^{1/2} \langle \nu_k^2 \rangle^{1/2} = c$ are

$$\lambda_k = \langle \lambda_k^2 \rangle^{1/2} = 1/R^o = 0.119935 \text{ m} \quad , \quad \nu_k = \langle \nu_k^2 \rangle^{1/2} = 2.49969 \times 10^9 \text{ Hz} \quad (41)$$

The classical definition of thermodynamic temperature based on two degrees of freedom

$$3kT' = < mv_x^2 > = 2m < v_{x+}^2 > \quad (42)$$

was recently modified to a new definition based on a single degree of freedom [32, 83]

$$3kT = m < v_{x+}^2 > \quad (43)$$

such that

$$T' = 2T \quad , \quad p' = 2p \quad (44)$$

The factor 2 in Eq. (44) results in the predicted speed of sound in air [32, 83]

$$a = v_{rmsx+} = \sqrt{p'/(2\rho)} = \sqrt{3kT'/(2m)} = \sqrt{3kT/m} \approx 357 \text{ m/s} \quad (45)$$

in close agreement with observations. Also, Eq. (45) leads to calculated root-mean-square molecular speeds (1346, 336, 360, 300, 952, 287) m/s that are in reasonable agreement with the observed velocities of sound (1286, 332, 337, 308, 972, 268) m/s in gases: (H_2, O_2, N_2, Ar, He, CO_2) [84].

The square root of 2 in Eq. (45) resolves the classical problem of *Newton* concerning his prediction of velocity of sound as

$$a = \sqrt{p' / \rho} \tag{46}$$

discussed by *Chandrasekhar* [85]

"Newton must have been baffled, not to say disappointed. Search as he might, he could find no flaw in his theoretical framework—neither could Euler, Lagrange, and Laplace; nor, indeed, anyone down to the present"

Indeed, predictions based on the expressions introduced by *Euler* $p = \rho v^2 / 3$, *Lagrange* $p \propto \rho^{4/3}$ as well as *Laplace*'s assumption of isentropic relation $p = \upsilon \rho^{\gamma}$, where υ is a constant and $\gamma = c_p / c_v$, that leads to the conventional expression for the speed of sound in ideal gas

$$a = \sqrt{\gamma RT} \tag{47}$$

are all found to deviate from the experimental data [32, 39].

The factor of 2 in Eq. (44) also leads to the modified value of *Joule-Mayer* mechanical equivalent of heat J introduced in [32, 83]

$$J = 2J_c = 2 \times 4.169 = 8338 \quad \text{J/kcal} \tag{48}$$

where the value $J_c = 4.169 \approx 4.17$ [kJ/kcal] is the average of the two values $J_c = (4.15, 4.19)$ reported by *Pauli* [86]. The number in Eq. (48) is thus identified as the universal gas constant in Eq. (39) when expressed in appropriate MKS system of units [32]

$$R^o = kN^o = J = 8338 \quad \text{J/(kmol.K)} \tag{49}$$

The modified value of the universal gas constant in Eq. (49) was recently identified [87] as *De Pretto* number 8338 that appeared in the mass–energy equivalence equation of *De Pretto* [88]

$$E = mc^2 \quad \text{Joules} \quad = mc^2 / 8338 \quad \text{kcal} \tag{50}$$

Unfortunately, the name of *Olinto De Pretto* in the history of evolution of mass energy equivalence is little known. Ironically, *Einstein's* best friend *Michele Besso* was a relative and close friend of *Olinto De Pretto's* brother *Augusto De Pretto*. The relativistic form of Eq. (50) was first introduced in 1900 by *Poincaré* [89]

$$E = m_r c^2 \tag{51}$$

where $m_r = m_o / \sqrt{1 - v^2 / c^2}$ is the *Lorentz* relativistic mass [83]. Since the expression (50) is the only equation in the paper by *De Pretto* [88], the exact method by which he arrived at the number 8338 is not known even though one possible method was recently suggested [87]. The important contributions by *Hasenöhrl* [90] and *Einstein* [91] as well as the principle of equivalence of the rest or gravitational mass and the inertial mass were discussed in a recent study [83].

V. Invariant Boltzmann Distribution Function

The kinetic theory of gas as introduced by *Maxwell* [36] and generalized by *Boltzmann* [37-38] is based on the nature of the molecular velocity distribution function that satisfies certain conditions of space isotropy and homogeneity and being stationary in time. However, in his work on generalization of *Maxwell's* result *Boltzmann* introduced the important concept of "complexions" and the associated combinatoric [41] that was subsequently used by *Planck* in his derivation of the equilibrium radiation spectrum [74, 75]. In the following the invariant model of statistical mechanics and *Boltzmann*'s combinatoric will be employed to arrive at *Boltzmann* distribution function.

To better reveal the generality of the concepts, rather than the usual scale of molecular-dynamics, we consider the statistical field of equilibrium eddy-dynamics EED at the scale β = e. According to Fig. 1, the homogenous isotropic turbulent field of EED is a hydrodynamic system h_h composed of an ensemble of fluid elements

$$\text{System}_{EED} = \text{Hydro} - \text{System} = h_\ell = \sum_k f_{k\ell} \tag{52}$$

Next, each fluid element f_k is an ensemble of a spectrum of eddies

$$\text{Element}_{EED} = \text{Fluid Element} = f_k = \sum_j e_{jk} \tag{53}$$

Finally, an eddy or the "atom" of EED field is by definition in Eq. (1) the most probable size of an ensemble of molecular clusters (Fig. 1)

$$\text{Atom}_{EED} = \text{Eddy} = e_j = \sum_i c_{ij} \tag{54}$$

At the lower scale of ECD each eddy of type j will correspond to the *energy level* j and is composed of ensemble of clusters or *quantum states* c_{ij} within the energy level j. Cluster of type c_{ij} does not refer to different cluster "specie" but rather to its different energy. The above procedure could then be extended to higher and lower scales within the hierarchy shown in Fig. 1

$$\text{System}_\beta = \text{Element}_{\beta+1} = \sum \text{Elements}_\beta \tag{55}$$

$$\text{Element}_\beta = \text{Atom}_{\beta+1} = \sum \text{Atom}_\beta \tag{56}$$

It is noted again that by Eq. (27) typical system size of (EED, ECD, EMD) scales are $(L_e, L_c, L_m) = (10^{-1}, 10^{-3}, 10^{-5})$ m.

Following *Boltzmann* [37, 38] and *Planck* [74] the number of complexions for distributing N_j indistinguishable eddies among g_j distinguishable cells or "quantum states" or eddy-clusters of ECD scale is [32]

$$W_j = \frac{(N_j + g_j - 1)!}{N_j!\,(g_j - 1)!} \tag{57}$$

The total number of complexions for system of independent energy levels W_j is obtained from Eq. (57) as

$$W = \prod_j \frac{(N_j + g_j - 1)!}{N_j!\,(g_j - 1)!} \tag{58}$$

As was discussed above, the hydrodynamic system is composed of g_j distinguishable fluid elements that are identified as energy levels of EED system. Each fluid element is considered to be composed of eddy clusters made of indistinguishable eddies. However, the smallest cluster contains only a single eddy and is therefore considered to be full since no other eddy can be added to this smallest cluster. Because an empty cluster has no physical significance, the total number of *available* cells or quantum states will be $(g_j - 1)$. Therefore, *Planck-Boltzmann* formula (57) is the exact probability of distribution of N_j indistinguishable oscillators (eddies) amongst $(g_j - 1)$ distinguishable *available* eddy clusters. The invariant model of statistical mechanics (Fig. 1) provides new perspectives on the probabilistic nature of Eq. (57) and the problem of distinguishability discussed by *Darrigol* [41].

Under the realistic assumptions

$$g_j \gg N_j \quad , \quad N_j \gg 1 \tag{59}$$

it is known that the number of complexions for *Bose-Einstein* statistics in Eq. (57) simplifies such that all three types namely "corrected" *Boltzmann*, *Bose-Einstein*, and *Fermi-Dirac* statistics will have [92]

$$W_j = g_j^{N_j} / N_j! \tag{60}$$

The most probable distribution is obtained by maximization of Eq. (60) that by *Sterling's* formula results in

$$\ln W_j = N_j \ln g_j - N_j \ln N_j + N_j \tag{61}$$

and hence

$$d(\ln W_j) = dN_j \ln(g_j / N_j) = 0 \tag{62}$$

In the sequel it will be argued that at thermodynamic equilibrium because of the equipartition principle of *Boltzmann* the energy of all levels U_j should be the same and equal to the most probable energy that defines the thermodynamic temperature such that

$$dU_j = \varepsilon_j dN_j + N_j d\varepsilon_j = 0 \tag{63}$$

or

$$dU_j = \varepsilon_j dN_j + \frac{N_j d\varepsilon_j}{dN_j} dN_j = \varepsilon_j dN_j + \frac{d(N_j \varepsilon_j)}{dN_j} dN_j$$
$$= \varepsilon_j dN_j + \left(\frac{dU_j}{dN_j}\right)_{S,V} dN_j = \varepsilon_j dN_j + \hat{\mu}_j dN_j \tag{64}$$

where *Gibbs* chemical potential is defined as

$$\hat{\mu}_j = \left(\frac{\partial U_j}{\partial N_j}\right)_{S,V,N_{i \neq j}} = \hat{g}_j = G_j / N_j \tag{65}$$

Introducing the *Lagrange* multipliers $-\beta'$ and α one obtains from equations (62) and (64)

$$dN_j \{\ln(g_j / N_j) - \beta'(\varepsilon_j - \alpha \hat{\mu}_j)\} = 0 \tag{66}$$

that leads to *Boltzmann* distribution

$$N_j = g_j e^{-\beta'(\varepsilon_j - \alpha\hat{\mu}_j)} = g_j e^{-(\varepsilon_j - \alpha\hat{\mu}_j)/kT} \tag{67}$$

It is emphasized that as opposed to the common practice in the above derivation the constant *Lagrange* multiplier α is separate and distinct from the chemical potential $\hat{\mu}_j$ that is a variable as required. Hence, for photons $\hat{\mu}_k = 0$ one can have $\alpha\hat{\mu}_k = 0$ without having to require $\alpha = 0$ corresponding to non-conservation of the number of photons. Following the classical methods [92-94] the first *Lagrange* multiplier becomes $\beta' = 1/kT$. In Sec. 10 it is shown that the second *Lagrange* multiplier is $\alpha = 1$.

VI. Invariant Planck Energy Distribution Function

In this section, the invariant *Planck* energy distribution law will be derived from the invariant *Boltzmann* statistics introduced in the previous section. To obtain a correspondence between photon gas at EKD scale and the kinetic theory of ideal gas in statistical fields of other scales, by equations (29)-(30) particles with the energy $\varepsilon_\beta = h_\beta \nu_\beta = h\nu_\beta = m_\beta v_\beta^2$ are viewed as *virtual oscillator* [72] that act as composite bosons [95] and hence follow *Bose-Einstein* statistics. It is well known that the maximization of the thermodynamic probability given by *Planck-Boltzmann* formula Eq. (57) leads directly to *Bose-Einstein* distribution [92-94]

$$N_j = \frac{g_j}{e^{\varepsilon_j / kT} - 1} \tag{68}$$

However, since *Boltzmann* distribution in Eq. (67) was derived by maximization of Eq. (57) as discussed in the previous section, it should also be possible to arrive at Eq. (68) directly from Eq. (67).

The analysis is first illustrated for the two consecutive equilibrium statistical fields of EED scale β = e when ("atom", cluster, system) are (eddy, fluid element, hydrodynamic system) identified by (e, f, h) with indices (j, k, h) and ECD at scale β = c when ("atom", element, system) are (molecule, cluster, eddy) identified by (m, c, e) with indices (m, i, j).

The statistical field of EED is a hydrodynamic system composed of a spectrum of fluid elements (energy levels) that are eddy clusters of various sizes as shown in Fig. 1. For an ideal gas at constant equilibrium temperature internal energy U_f will be constant and by Eq. (65) one sets $\hat{\mu}_f = 0$ and the number of fluid element of type f (energy level f) in the hydrodynamic system from Eq. (67) becomes

$$N_{fh} = g_{fh} e^{-\varepsilon_f/kT} \quad (69)$$

Assuming that the degeneracy of all levels f is identical to a constant average value $g_{fh} = \overline{g}_{fh}$, the average number of fluid elements in the hydrodynamic system h from Eq. (69) becomes

$$\overline{N}_{fh} = \sum_f g_{fh} e^{-\varepsilon_f/kT} = \overline{g}_{fh} \sum_f e^{-\varepsilon_f/kT}$$
$$= \overline{g}_{fh} \sum_j e^{-N_j \varepsilon_j/kT} = \overline{g}_{fh} \sum_j \left(e^{-\varepsilon_j/kT}\right)^{N_j}$$
$$= \frac{\overline{g}_{fh}}{1 - e^{-\varepsilon_j/kT}} \quad (70)$$

In the derivation of Eq. (70) the relation

$$\varepsilon_f = \sum_j \varepsilon_{jf} = N_j \varepsilon_{jf} = U_j \quad (71)$$

for the internal energy of the fluid element f has been employed that is based on the assumption that all eddies of energy level f are at equilibrium and therefore stochastically stationary indistinguishable eddies with stationary size and energy.

At the next lower scale of ECD, the system is a fluid element composed of a spectrum of eddies that are energy levels of ECD field. Eddies themselves are composed of a spectrum of molecular clusters i.e. cluster of molecular-clusters hence super-cluster. Again, following the classical methods of *Boltzmann* [92-94], for an ideal gas at constant temperature hence U_j by Eq. (65) $\hat{\mu}_j = 0$ and from Eq. (67) the number of eddies in the energy level j within the fluid element f becomes

$$N_{jf} = g_{jf} e^{-\varepsilon_j/kT} \quad (72)$$

The result in Eq. (72) is based on the fact that all eddies of element jf are considered to be indistinguishable with identical energy

$$\varepsilon_j = \sum_i \varepsilon_{ij} = N_i \varepsilon_{ij} = U_i \quad (73)$$

that is in harmony with Eq. (71).

It is now possible to determine the distribution of eddies as *Planck oscillators* (*Heisenberg-Kramers virtual oscillators)* among various energy levels (fluid elements) with degeneracy under the constraint of a constant energy of all levels in Eq. (63). From equations (70)-(72), the average number of eddies in the energy level f of hydrodynamic system can be expressed as

$$N_j = \overline{N}_{jh} = \overline{N}_{fh} \overline{N}_{jf} = \frac{g_j}{e^{\varepsilon_j/kT} - 1} \quad (74)$$

that is *Bose-Einstein* distribution in Eq. (68) when the total degeneracy is defined as $g_j = \overline{g}_{jh} = \overline{g}_{jf} \overline{g}_{fh}$.

In the sequel it will be shown that the relevant degeneracy g_e for ideal gas at equilibrium is similar to the classical *Rayleigh-Jeans* [96-97] expression for degeneracy of equilibrium radiation here expressed as

$$dg_j = \frac{8\pi V}{u_j^3} \nu_j^2 d\nu_j \quad (75)$$

At thermal equilibrium Eq. (75) denotes the number of eddies (oscillators) at constant mean "atomic" velocity u_j in a hydrodynamic system with volume *V* within the frequency interval ν_j to $\nu_j + d\nu_j$. The results in equations (74) and (75) lead to *Planck* [34, 74] energy distribution function for isotropic turbulence at EED scale

$$\frac{\varepsilon_j dN_j}{V} = \frac{8\pi h}{u_j^3} \frac{\nu_j^3}{e^{h\nu_j/kT} - 1} d\nu_j \quad (76)$$

when the energy of each eddy is $\varepsilon_j = h\nu_j$. The calculated energy distribution from Eq. (76) at T = 300 K is shown in Fig. 6.

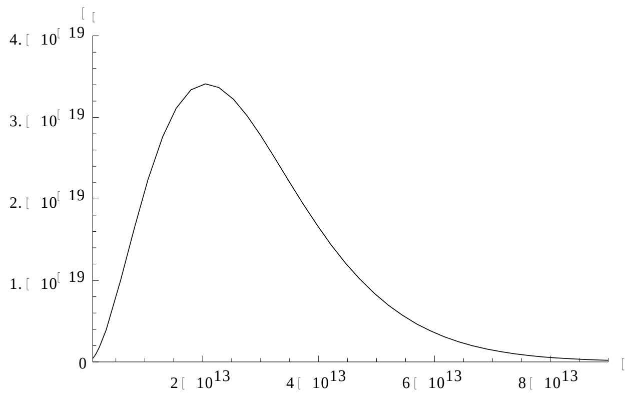


Fig. 6 Planck energy distribution law governing the energy spectrum of eddies at the temperature T = 300 K.

The three-dimensional energy spectrum E(k) for isotropic turbulence measured by *Van Atta* and *Chen* [98-99] and shown in Fig. 7 is in qualitative agreement with *Planck* energy spectrum shown in Fig. 6 [34].

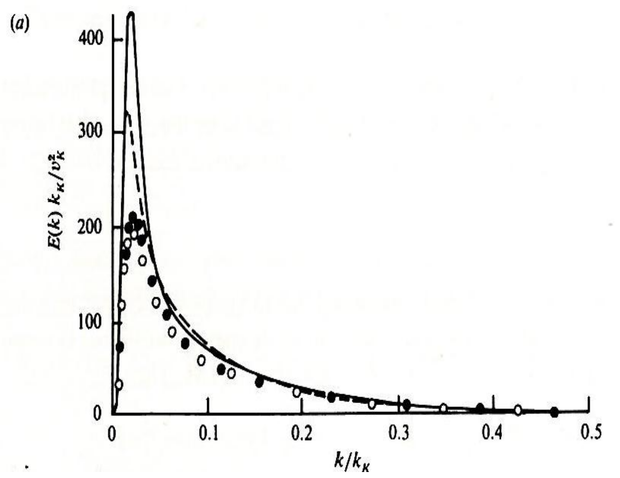


Fig. 7 Normalized three-dimensional energy spectra for isotropic turbulence [98].

In fact, it is expected that to maintain stationary isotropic turbulence both energy supply as well as energy dissipation spectrum should follow *Planck* law in Eq. (76). The experimental data [100] obtained for one dimensional dissipation spectrum along with *Planck* energy distribution as well as this same distribution shifted by a constant amount of energy are shown in Fig. 8.

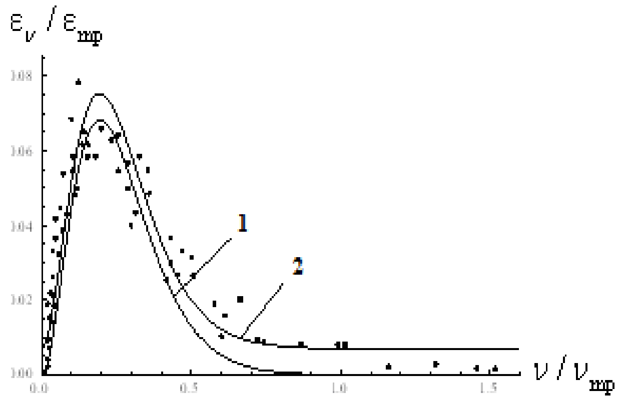


Fig. 8 One-dimensional dissipation spectrum [100] compared with (1) Planck energy distribution (2) Planck energy distribution with constant displacement.

Similar comparison with *Planck* energy distribution as shown in Fig. 8 is obtained with the experimental data for one-dimensional dissipation spectrum of isotropic turbulence shown in Fig. 9 from the study of *Saddoughi and Veeravalli* [101]

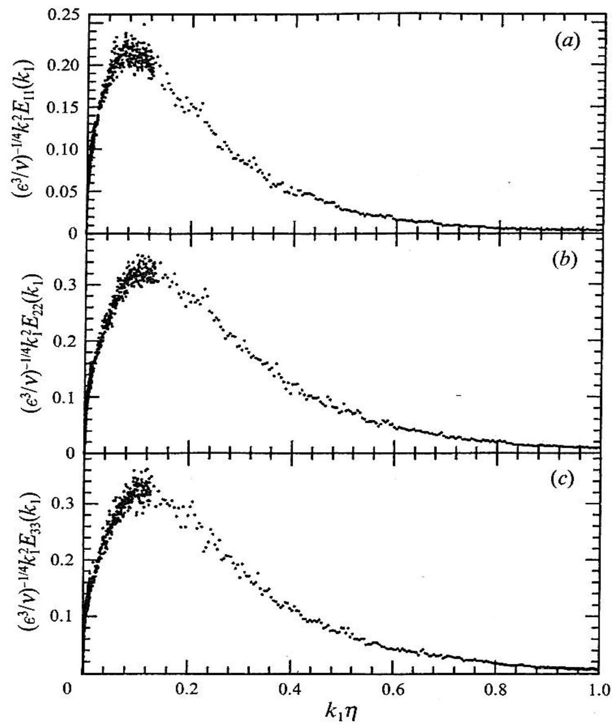


Fig. 9 One-dimensional dissipation spectra (a) u_1-spectrum (b) u_2-spectrum (c) u_3-spectrum [101].

In a more recent experimental investigation the energy spectrum of turbulent flow within the boundary layer in close vicinity of rigid wall was measured by *Marusic et al.* [102] and the reported energy spectrum shown in Fig. 10 appear to have profiles quite similar to *Planck* distribution law.

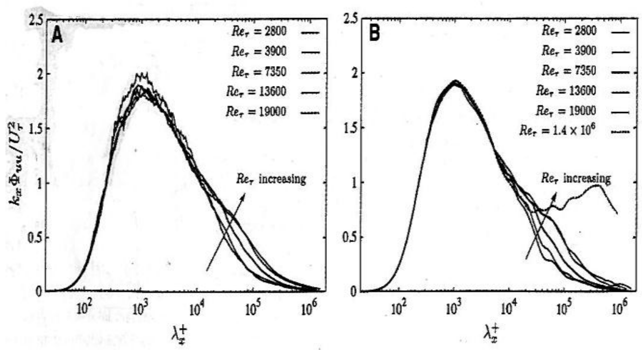


Fig. 10 Reynolds number evolution of the pre-multiplied energy spectra of stream wise velocity at the inner-peak location (z+ = 15) for the true measurements (A) and the prediction based on the filtered u signal measured in the log region (B) [102].

Also, the normalized three-dimensional energy spectrum for homogeneous isotropic turbulent field was obtained from the transformation of one-dimensional energy spectrum of *Lin* [103] by *Ling and Huang* [104] as

$$E^* = \frac{\alpha^{*2}}{3}(K^* + \alpha^* K^{*2})\ \exp(-\alpha^* K^*) \qquad (77)$$

with the distribution comparable with Fig. 6.

A most important aspect of *Planck* law is that at a given fixed temperature the energy spectrum of equilibrium field is time invariant. Since one may view *Planck* distribution as energy spectrum of eddy cluster sizes this means that cluster sizes are stationary. Therefore, even though the number of eddies N_{jf} and their energy ε_{jf} in different fluid elements (energy levels) are different their product that is the total energy of all energy levels is the same

$$U_j = \sum_j \varepsilon_j = N_j \varepsilon_j = U_{j+1} = ... = U_{mpj} = \overline{U} \tag{78}$$

in accordance with Eq. (63). Thus *Boltzmann's* equipartition principle is satisfied in order to maintain time independent spectrum (Fig. 6) and avoid *Maxwell's* demon paradox [35]. Therefore, in stationary isotropic turbulence, energy flux occurs between fluid elements by transition of eddies of diverse sizes while leaving the fluid elements stochastically stationary in time. A schematic diagram of energy flux across hierarchies of eddies from large to small size is shown in Fig. 11 from the study by *Lumley et al.* [105].

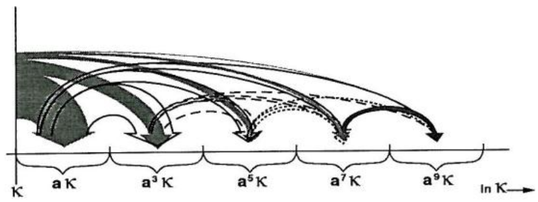


Fig. 11 A realistic view of spectral energy flux [105].

In Sec. 11, it will be suggested that the exchange of eddies between various size fluid elements (energy levels) is governed by quantum mechanics through an invariant *Schrödinger* equation (206). Therefore, transition of an eddy from a small rapidly oscillating fluid element to a large slowly oscillating fluid element results in energy emission by "subatomic particle" that for EED will be a molecular cluster c_{ji} as schematically shown in Fig. 12.

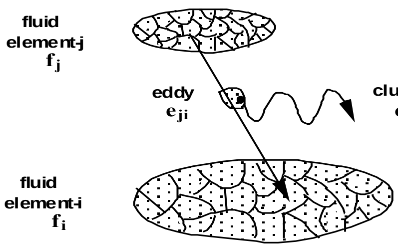


Fig. 12 Transition of eddy e_{ij} from fluid element-j to fluid element-i leading to emission of cluster c_{ij}.

Hence, the stochastically stationary states of fluid elements are due to energy exchange through transitions of eddies according to

$$\Delta\varepsilon_{ji\beta} = \varepsilon_{j\beta} - \varepsilon_{i\beta} = h(\nu_{j\beta} - \nu_{i\beta}) \tag{79}$$

parallel to *Bohr's* stationary states in atomic theory [72] to be further discussed in Sec. 11.

The above procedures in equations (68)-(76) could be applied to other pairs of adjacent statistical fields (ECD-EMD), (EMD-EAD), (EAD-ESD), (ESD-EKD), (EKD-ETD), … shown in Fig. 1 leading to *Planck* energy distribution function for the energy spectrum of respectively molecular-clusters, molecules, atoms, sub-particles (electrons), photons, tachyons, . . . at thermodynamic equilibrium. Therefore, Eq. (76) is the *invariant Planck energy distribution law* and can be written in invariant form for any scale β as [34]

$$\frac{\varepsilon_\beta dN_\beta}{V} = \frac{8\pi h}{u_\beta^3} \frac{\nu_\beta^3}{e^{h\nu_\beta/kT} - 1} d\nu_\beta \tag{80}$$

with the spectrum shown in Fig. 6.

The invariant *Planck* energy distribution in Eq. (80) is a universal law giving energy spectra of all equilibrium statistical fields from cosmic to sub-photonic scales shown in Fig. 1. Such universality is evidenced by the fact that the measured deviation of *Penzias-Wilson* cosmic background radiation temperature of about 2.73 K from *Planck* law is about 10^{-5} K. In view of the finite gravitational mass of photon in Eq. (37), it is expected that as the temperature of the radiation field is sufficiently lowered photon condensation should occur parallel to superconductivity, BEC, and superfluidity at the scales of electro-dynamics, atomic-dynamics, and molecular-dynamics [51]. Such phenomena have indeed been observed in a recent study [106] reporting on light condensation and formation of photon droplets. Furthermore, one expects a hierarchy of condensation phenomena to continue to tachyonic [107], or sub-tachyonic fields … ad infinitum.

The important scales ESD β = s and EKD β = k are respectively associated with the fields of stochastic electrodynamics SED and stochastic chromo-dynamics SCD [1-17]. For EKD scale of photon gas β = k, also identified as *Casimir* [73] vacuum or the physical space with the most probable thermal speed of photon in vacuum $u_k = v_{mpt} = c$ [83], the result in Eq. (80) corresponds to a spectrum of photon clusters with energy distribution given by the classical *Planck* energy distribution law [74]

$$\frac{\varepsilon_\nu dN_\nu}{V} = \frac{8\pi h}{c^3} \frac{\nu^3}{e^{h\nu/kT} - 1} d\nu \tag{81}$$

The notion of "*molecules of light*" as clusters of photons is in accordance with the perceptions of *de Broglie* [41, 108, 109]. It is emphasized that the velocity of light is therefore a function of the temperature of *Casimir* [73] vacuum, i.e. the tachyonic fluid [83] that is *Dirac* [110] stochastic ether or *de Broglie* [3] hidden thermostat. However since such vacuum temperature changes by expansion of the cosmos through eons [35], one may assume that c is nearly a constant for the time durations relevant to human civilization.

The historical evolution of *Planck* law of equilibrium radiation, his spectral energy distribution function (81), and the central role of energy quanta ε = hν are all intimately related to the statistical mechanics of *Boltzmann* discussed in the previous section. This is most evident from the following quotation taken from the important 1872 paper of *Boltzmann* [37, 39]

"We wish to replace the continuous variable x by a series of discrete values ε, 2ε, 3ε … pε. Hence we must assume that our molecules are not able to take up a continuous series of kinetic energy values, but rather only values that are multiples of a certain quantity ε. Otherwise we shall treat exactly the same problem as before. We have many gas molecules in a space R. They are able to have only the following kinetic energies:

ε, 2ε, 3ε, 4ε, . . . pε.

No molecule may have an intermediate or greater energy. When two molecules collide, they can change their kinetic energies in many different ways. However, after the collision the kinetic energy of each molecule must always be a multiple of ε. I certainly do not need to remark that for the moment we are not concerned with a real physical problem. It would be difficult to imagine an apparatus that could regulate the collisions of two bodies in such a way that their kinetic energies after a collision are always multiples of ε. That is not a question here. "

The quotation given above and the introduction of the statistical mechanics of complexions discussed in the previous section are testimony to the significant role played by *Boltzmann* in the development of the foundation of quantum mechanics as was also emphasized by *Planck* in his *Nobel* lecture [61, 111].

Similarly, *Boltzmann* gas theory had a strong influence on *Einstein* in the development of the theory of *Brownian* motion even though *Boltzmann* himself made only a brief passing remark about the phenomena [61]

". . . likewise, it is observed that very small particles in a gas execute motions which result from the fact that the pressure on the surface of the particles may fluctuate."

Although *Einstein* did not mention the importance of *Boltzmann's* gas theory in his autobiographical sketch [61]

"Not acquainted with the earlier investigations of Boltzmann and Gibbs which appeared earlier and which actually exhausted the subject, I developed the statistical mechanics and the molecular kinetic theory of thermodynamics which was based on the former. My major aim in this was to find facts which would guarantee as much as possible the existence of atoms of definite finite size. In the midst of this I discovered that, according to atomic theory, there would have to be a movement of suspended microscopic particles open to observation, without knowing that observations concerning Brownian motion were long familiar"

much earlier in September of 1900 *Einstein* did praise *Boltzmann's* work in a letter to *Mileva* [61, 112]

"The Boltzmann is magnificent. I have almost finished it. He is a masterly expounder. I am firmly convinced that the principles of the theory are right, which means that I am convinced that in the case of gases we are really dealing with discrete mass points of definite size, which are moving according to certain conditions. Boltzmann very correctly emphasizes that the hypothetical forces between the molecules are not an essential component of the theory, as the whole energy is of the kinetic kind. This is a step forward in the dynamical explanation of physical phenomena"

Similar high praise of *Boltzmann's* theory appeared in April 1901 letter of *Einstein* to *Mileva* [112]

"I am presently studying Boltzmann's gas theory again. It is all very good, but not enough emphasis is placed on a comparison with reality. But I think that there is enough empirical material for our investigation in the O. E. Meyer. You can check it the next time you are in the library. But this can wait until I get back from Switzerland. In general, I think this book deserves to be studied more carefully."

The central role of *Boltzmann* in *Einstein's* work on statistical mechanics has also been recently emphasized by *Renn* [113]

"In this work I argue that statistical mechanics, at least in the version published by Einstein in 1902 (Einstein 1902b), was the result of a reinterpretation of already existing results by Boltzmann."

In order to better reveal the nature of particles versus the background fields at (ESD-EKD) and (EKD-ETD) scales, we examine the normalized *Maxwell-Boltzmann* speed distribution in Eq. (119) from Sec. 8 shown in Fig. 13.

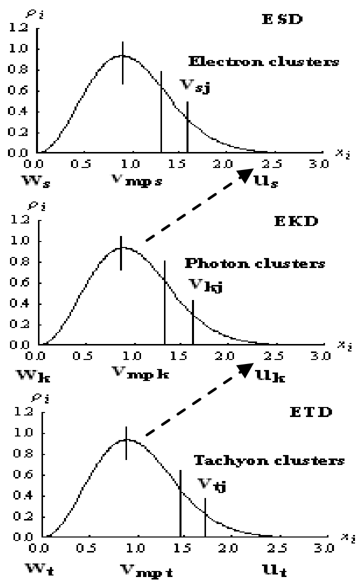


Fig. 13 Maxwell-Boltzmann speed distribution for ESD, EKD, and ETD fields.

According to Fig. 13, in ETD field one starts with tachyon [107] "atom" to form a spectrum of tachyon clusters. Next, photon or *de Broglie* "atom of light" [108] is defined as the most probable size tachyon cluster of the stationary ETD field (Fig. 13). Moving to the next larger scale of EKD, one forms a spectrum of photon clusters representing ideal photon gas of equilibrium radiation field. Finally, one identifies the "electron" as the most probable size photon cluster (Fig. 13) of stationary EKD field. From ratio of the masses of electron and photon in Eq. (37) the number of photons in an electron is estimated as

$$N_{ke^-} = \frac{9.1086\times10^{-31}}{1.84278\times10^{-41}} = 4.9428\times10^{10} \quad \text{Photons} \tag{82}$$

The above definition of electron suggests that not all electrons may be exactly identical since by Eq. (82) a change of few hundred photons may not be experimentally detectable due to small photon mass.

With electron defined as the "atom" of electrodynamics, one constructs a spectrum of electron clusters to form the statistical field of equilibrium sub-particle dynamics ESD (SED) as ideal electron gas in harmony with the perceptions of *Lorentz* [114]

"Now, if within an electron there is ether, there can also be an electromagnetic field, and all we have got to do is to establish a system of equations that may be applied as well to the parts of the ether where there is an electric charge, i.e. to the electrons, as to those where there is none."

The most probable electron cluster of ESD field is next identified as the "atom" of EAD field. As shown in Fig. 13, the most probable element of scale β becomes the "atom" of the higher scale β+1 and the "system" of lower scale β−1

$$w_\beta = v_{mp\beta+1} = u_{\beta+2} \tag{83}$$

in accordance with equations (1)-(2).

At EKD scale *Planck* law (81) gives energy spectrum of photon conglomerates, *Sackur's* "clusters", or *Planck's* "quantum sphere of action" as described by *Darrigol* [41] with sizes given by *Maxwell-Boltzmann* distribution (Fig. 13) in harmony with the perceptions of *de Broglie* [109]

"Existence of conglomerations of atoms of light whose movements are not independent but coherent"

Thus photon is identified as the most probable size tachyon cluster (Fig. 13) of stationary ETD field. From ratio of the masses of photon in Eq. (37) and tachyon $m_t = m_g = 3.08\times10^{-69}$ kg [83, 115] the number of tachyons in a photon is estimated as

$$N_{tk} = \frac{1.84278\times10^{-41}}{3.08\times10^{-69}} = 5.983\times10^{27} \quad \text{Tachyons} \tag{84}$$

Comparison of equations (82) and (84) suggests that there may be another particle (perhaps *Pauli's* neutrino) with the approximate mass of $m_\nu \approx 10^{-55}$ kg between photon and tachyon scales. Also, as stated earlier, the "atoms" of all statistical fields shown in Figs. 1, 4, and 13 are considered to be "composite bosons" [95] made of "*Cooper pairs*" of the most probable size cluster of the statistical field of the adjacent lower scale (Fig. 13). Indeed, according to *de Broglie* as emphasized by *Lochak* [109],

"Photon cannot be an elementary particle and must be composed of a pair of particles with small mass, maybe "neutrinos"."

Therefore, one expects another statistical field called equilibrium neutrino-dynamics END to separate EKD and ETD fields shown in Figs. 1 and 13.

The invariant *Planck* law in Eq. (80) leads to the invariant *Wien* [93] displacement law

$$\lambda_{w\beta}T = 0.2014c_2 \tag{85}$$

For β = k by equations (29) and (30) the second radiation constant c_2 is identified as the inverse square of the universal gas constant in Eq. (39)

$$c_2 = \frac{hc}{k} = \frac{hkc}{k^2} = \frac{m_k^2c^4}{k^2} = \frac{1}{k^2N^{o2}} = \frac{1}{R^{o2}} \tag{86}$$

such that one may also express Eq. (85) as

$$\lambda_{m\beta}T = \frac{0.2014}{R^{o2}} = 0.002897 \text{ m-deg} \tag{87}$$

It is also possible to express Eq. (87) in terms of the root mean square wavelength of photons in vacuum

$$c_2 = \frac{hc}{k} = \frac{m_k\lambda_k cc}{m_k\nu_k c} = \frac{\lambda_k c}{\nu_k} = \lambda_k^2 \tag{88}$$

from Eq. (41)

$$\lambda_k = 0.119933 \quad \text{m} \tag{89}$$

By the definition of *Boltzmann* constant in equations (31b), (33), and (36) the absolute thermodynamic temperature becomes the root mean square wavelength of the most probable state

$$T = <\lambda_{w\beta}^2>^{1/2} = \lambda_{mp\beta} \tag{90}$$

Therefore, by equations (87) and (89) *Wien* displacement law in Eq. (85) may be also expressed as

$$\lambda_{wk}T = \lambda_{mp,k}^2 = 0.2014\lambda_k^2 \tag{91}$$

relating the most probable and the root mean square wavelengths of photons in radiation field at thermodynamic equilibrium.

It is possible to introduce a displacement law for most probable frequency parallel to *Wien's* displacement law for most probable wavelength in Eq. (85). By setting the derivative of *Planck* energy density to zero one arrives at the transcendental equation for maximum frequency as

$$e^{c_2 \nu_w / kcT} + \frac{c_2 \nu_w}{3kcT} - 1 = 0 \tag{92}$$

From the numerical solution of Eq. (92) one obtains the *frequency displacement law*

$$\frac{T}{\nu_w} = 0.354428 \frac{c_2}{c} = 0.354428 \frac{h}{k} \tag{93}$$

or

$$\nu_w = 5.8807375 \times 10^5 T \tag{94}$$

From Eq. (94) one obtains the frequency at the maxima of *Planck* energy distribution at temperature T such as $\nu_w = 1.764 \times 10^{13}$ Hz at T = 300K in accordance with Fig. 6 and $\nu_w = 3.5284 \times 10^{14}$ Hz at T = 6000K in agreement with Fig. 6.1 of *Baierlein* [94].

From division of *Wien* displacement laws for wavelength in Eq. (85) and frequency in Eq. (93) one obtains

$$\lambda_{wk} \nu_{wk} = 0.5682 \text{ c} \tag{95}$$

Because by Eq. (41) the speed of light in vacuum is

$$\lambda_k \nu_k = c \tag{96}$$

one can express Eq. (95) as

$$\frac{\lambda_{mk} \nu_{mk}}{\lambda_k \nu_k} = \frac{v_{mp,k}}{v_{r.m.s,k}} = \frac{v_{mk}}{c} \approx 0.57 \tag{97}$$

The result in Eq. (97) may be compared with the ratio of the most probable speed $v_{mp} = \sqrt{kT/m}$ and the root mean square speed $v_{r.m.s} = \sqrt{3kT/m}$ that is

$$v_{mp} / v_{r.m.s} = 1/\sqrt{3} \approx 0.577 \tag{98}$$

The reason for the difference between equations (97) and (98) requires further future examination.

At thermodynamic equilibrium each system of Fig. 13 will be stationary at a given constant temperature. The total energy of such equilibrium field will be the sum of the potential and internal energy expressed by the modified form of the first law of thermodynamics [31] to be further discussed in Sec. 10

$$Q_\beta = H_\beta = U_\beta + p_\beta V_\beta \tag{99}$$

In a recent investigation [83] it was shown that for monatomic ideal gas with $\tilde{c}_v = 3R^o$ and $\tilde{c}_p = 4R^o$ one may express Eq. (99) as [32]

$$Q_\beta = U_\beta + PV_\beta = H_\beta = \frac{3}{4} H_\beta + \frac{1}{4} H_\beta = E_{de\beta} + E_{dm\beta} \tag{100}$$

Therefore, the total energy (mass) of the atom of β scale is the sum of the internal energy (dark energy $DE_{\beta-1}$) and potential energy (dark matter $DM_{\beta-1}$) at the lower scale β−1 [32, 83]

$$\varepsilon_\beta = E_{\beta-1} = U_{\beta-1} + p_{\beta-1} V_{\beta-1} = DE_{\beta-1} + DM_{\beta-1} \tag{101}$$

To better reveal the origin of the potential energy $p_{\beta-1} V_{\beta-1}$ in Eq. (101) one notes that by Eq. (30) the dimensionless particle energy in *Maxwell-Boltzmann* distribution in Eq. (111) could be expressed as

$$\frac{mv_j^2}{kT} = \frac{mv_j^2}{mv_{mp}^2} = \frac{h\nu_j}{h\nu_{mp}} = \frac{\nu_j}{\nu_{mp}} \tag{102}$$

that by $v_j = \lambda_j \nu_j$ gives

$$v_j / v_{mp} = (\lambda_j / \lambda_{mp})^{-1} \tag{103}$$

Therefore *Maxwell-Boltzmann* distribution in Eq. (111) may be expressed as a function of inverse of dimensionless wavelength by Eq. (103) thus revealing the relative (atomic, element, and system) lengths $(\ell, \lambda, L)_\beta = (0, 1, \infty)_\beta$ of the adjacent scales β and β−1 as shown in Fig. 14.

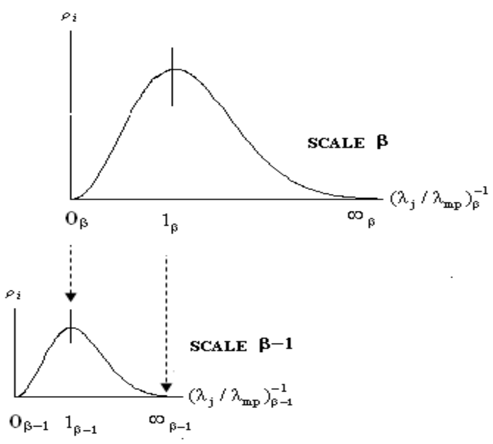


Fig. 14 Maxwell-Boltzmann speed distribution as a function of oscillator wavelengths $(\lambda_j/\lambda_{mp})^{-1}$.

According to Fig. 14, the interval $(0, 1)_\beta$ of scale β becomes $(1, \infty)_{\beta-1}$ of β−1 scale. However, the interval $(0, 1)_{\beta-1}$ is only revealed at $\beta-1$ scale and is unobservable at the larger scale β (Figs. 14, 18). Therefore, in three dimensions such coordinate extensions results in volume generation leading to release of potential energy as schematically shown in Fig. 15.

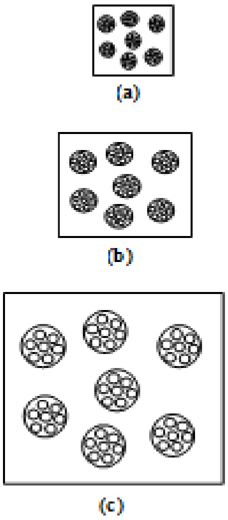


Fig. 15 Effects of internal versus external potential energy as the system volume is increased.

Hence by equations (102)-(103) as one decompactifies the atom of scale β, ¾ of the total mass (energy) of the atom "*evaporates*" into energy due to *internal* translational, rotational, and vibrational (pulsational) motions and is therefore none-baryonic and defined as dark energy (electromagnetic mass) [83]. The remaining ¼ of the total mass that appears as potential energy (dark matter), [83] in part "evaporates" as new volume generation (Figs. 15, 18) and in part forms the gravitational mass (dark matter) of the next lower scale [32, 83]

$$DM_{\beta-1} = E_{\beta-2} = DE_{\beta-2} + DM_{\beta-2} \tag{104}$$

The concepts of internal versus external potential energy are further discussed in the following section.

As an example, to determine the total energy of a photon one starts from the thermodynamic relation for an ideal photon gas

$$\tilde{h} = \tilde{u} + p\tilde{v} \tag{105}$$

with specific molar enthalpy, internal energy and volume $(\tilde{h}, \tilde{u}, \tilde{v}) = (H, U, V)/\tilde{N}$ that can also be expressed as

$$\tilde{c}_p = \tilde{c}_v + R^o \tag{106}$$

where $\tilde{N} = N/N^o$, and $\tilde{W} = N^o m$ is the molecular weight. Since *Poisson* coefficient γ of photon gas is $\gamma = \tilde{c}_p / \tilde{c}_v = 4/3$, one arrives at $\tilde{c}_p = 4R^o$ and $\tilde{c}_v = 3R^o$ such that by Eq. (105) the total energy of the photon could be expressed as

$$E_\gamma = (3/4)m_k c^2 + (1/4)m_k c^2 = E_{de\gamma} + E_{dm\gamma} \tag{107}$$

From Eq. (107) one concludes that of the total energy constituting a photon, ¾ is associated with the electromagnetic field (dark energy E_{de}) and ¼ with the gravitational field (dark matter E_{dm}) [32, 83]. Hence, as one decompactifies atoms of smaller and smaller scales by Eqs. (38), (40), and (100)-(107) ultimately all matter will be composed of dark energy or electromagnetic mass as was anticipated by both *Lorentz* [116] and *Poincaré* [117-119].

It is known that exactly ¾ and ¼ of the total energy of *Planck* black body equilibrium radiation falls on $\lambda > \lambda_w$ and $\lambda < \lambda_w$ sides of λ_w given by the *Wien* displacement law in Eq. (85). Indeed, the first part of Eq. (107) confirms the apparent mass $\mu_H = 4E/3c^2$ that was measured for the black body radiation pressure in the pioneering experiments by *Hasenöhrl* [90] in 1905. According to Eq. (107), the finite gravitational mass of the photon in Eq. (37) that is associated with *Poincaré* [117-119] stress accounts for the remaining ¼ of the total mass as dark matter. This longitudinal component would be absent if photon gravitational mass were zero in harmony with the perceptions of *Higgs* [120]. The result in Eq. (107) is also consistent with the general theory of relativity of *Einstein* [121] according to which of the total energy constituting matter ¾ is to be ascribed to the electromagnetic field and ¼ to the gravitational field.

VII. Invariant Maxwell-Boltzmann Speed Distribution Function

Because of its definition, the energy spectrum of particles in an equilibrium statistical field is expected to be closely connected to the spectrum of speeds of particles. Indeed, it is possible to obtain the invariant *Maxwell-Boltzmann* distribution function directly from the invariant *Planck* distribution function in Eq. (80) that in view of equations (34) and (37) can be written as

$$dN_{\nu\beta} = \frac{8\pi V}{u_\beta^3} \frac{\nu_\beta^2 d\nu_\beta}{e^{\varepsilon_\beta/kT} - 1} = \frac{8\pi V\, m_\beta^3}{u_\beta^3 m_\beta^3 \lambda_\beta^3} \frac{\lambda_\beta^2 \nu_\beta^2 \lambda_\beta d\nu_\beta}{e^{\varepsilon_\beta/kT} - 1}$$

$$= \frac{8\pi m_\beta^3 V}{h^3} \frac{u_\beta^2 du_\beta e^{-\varepsilon_\beta/kT}}{1 - e^{-\varepsilon_\beta/kT}} \tag{108}$$

Substituting for the partition function $Z = N = \sum g_{mc} e^{-\varepsilon_{cj}/kT} = g_{mc}/(1 - e^{-\varepsilon_\beta/kT})$ from Eq. (70), and the degeneracy for speed $g_{s\beta} = \overline{g}_{mc\beta}$

$$g_{s\beta} = 2V[(2\pi m_\beta kT)/h^2]^{3/2} \tag{109}$$

obtained from the normalization condition

$$\int_0^\infty dN_{\nu\beta}/N = 1 \tag{110}$$

into Eq. (108) results in the invariant *Maxwell-Boltzmann* speed distribution function

$$\frac{dN_{u\beta}}{N} = 4\pi \left(\frac{m_\beta}{2\pi k T_\beta}\right)^{3/2} u_\beta^2 \, e^{-\varepsilon_\beta / kT_\beta} du_\beta \qquad (111)$$

By Eq. (111), one arrives at a hierarchy of embedded *Maxwell-Boltzmann* distribution functions for EED, ECD, and EMD scales shown in Fig. 16.

As stated earlier, the invariant results in equations (80) and (111) suggest that particles of all statistical fields (Fig. 1) will have *Gaussian* velocity distribution, *Planck* energy distribution, and *Maxwell-Boltzmann* speed distribution.

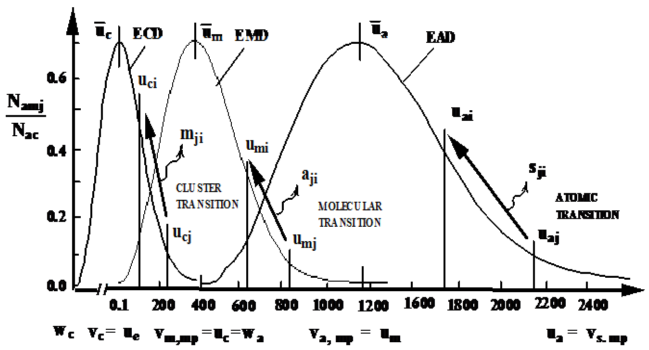


Fig. 16 Maxwell-Boltzmann speed distribution viewed as stationary spectra of cluster sizes for EED, ECD, and EMD scales at 300 K [32].

It is possible to express the number of degeneracy commonly obtained from field quantization [92-94] for particles, *Heisenberg-Kramers* [72] virtual oscillators in a spherical volume V_s as

$$g_\beta = 2V_S / \lambda_\beta^3 \qquad (112)$$

where λ_β^3 is the *rectangular* volume occupied by each oscillator $V_o = \lambda_\beta^3$ when due to isotropy $\lambda_\beta = <\lambda_{x\beta}^2>^{1/2} = <\lambda_{y\beta}^2>^{1/2} = <\lambda_{z\beta}^2>^{1/2}$ and the factor 2 comes from allowing particles to have two modes either (up) or (down) iso-spin (polarization). The system spherical V_S and rectangular V volumes are related as

$$V_S = \frac{4\pi}{3} R^3 = \frac{4\pi}{3} V \qquad (113)$$

For systems in thermodynamic equilibrium the temperature $3kT_\beta = m_\beta <u_\beta^2>$ will be constant and hence $<u_\beta^2>^{1/2} = <\lambda_\beta^2>^{1/2} <\nu_\beta^2>^{1/2}$ or

$$\lambda_\beta = u_\beta / \nu_\beta \qquad (114)$$

Substituting from equations (113)-(114) into Eq. (112) results in

$$g_\beta = \frac{8\pi V}{3u_\beta^3} \nu_\beta^3 \qquad (115)$$

that leads to the number of oscillators between frequencies ν_β and $\nu_\beta + d\nu_\beta$

$$dg_\beta = \frac{8\pi V}{u_\beta^3} \nu_\beta^2 d\nu_\beta \qquad (116)$$

in accordance with *Rayleigh-Jeans* expression in Eq. (75).

The expression in Eq. (116) for degeneracy is for application to *Planck* law involving frequency as the variable for energy quanta ε = hν. It is also possible to arrive at the degeneracy in Eq. (109) for *Maxwell-Boltzmann* speed distribution from Eq. (112). However, because only positive values of speeds (u_x, u_y, u_z) are allowed one must take 1/8 of the total volume of the velocity space and Eq. (112) in terms of the relevant volume gives

$$g_{s\beta} = 2V/\lambda_\beta^3 = \frac{2(L')^3}{\lambda_\beta^3} = \frac{2(2L'/\sqrt{\pi})^3 \pi^{3/2}}{8\lambda_\beta^3}$$

$$= \frac{2(2L'/\sqrt{\pi})^3 (\pi m_\beta^2 u_\beta^2)^{3/2}}{8\lambda_\beta^3 m_\beta^3 u_\beta^3} = \frac{2L^3 (\pi m_\beta^2 u_\beta^2)^{3/2}}{8h_\beta^3}$$

$$= 2V_c \left(\frac{\pi m_\beta k T_\beta}{h_\beta^2}\right)^{3/2} = 2\overline{V}\left(\frac{2\pi m_\beta k T_\beta}{h^2}\right)^{3/2} \qquad (117)$$

that is in accordance with Eq. (109). In Eq. (117) the correct relevant volume of the speed space is $V_c = (2L'/\sqrt{\pi})^3/8$, while $\overline{V} = (L'/\sqrt{2\pi})^3$, and $h_\beta = h$ by Eq. (36). The coordinate L' in Eq. (117) was first normalized as $L = 2L'/\sqrt{\pi}$ with a *measure* based on *Gauss's* error function as discussed in Ref. 122 and shown in Fig. 18. The result in Eq. (117) is twice the classical translational degeneracy [92]

$$g_t = V\left(\frac{2\pi m k T}{h^2}\right)^{3/2} \qquad (118)$$

The additional factor of two arises from the fact that similar to *Boltzmann* factor $e^{-h\nu/kT}$ in *Planck* distribution law in Eq. (81) by equations (42)-(44) the modified *Maxwell-Boltzmann* distribution in Eq. (111) will also involve $e^{-\varepsilon_\beta/kT} = e^{-m v_\beta^2/kT}$ rather than the classical expression $e^{-mv^2/2kT}$.

VIII. Connections between Riemann Hypothesis and Normalized Maxwell-Boltzmann Distribution Function

Because *Maxwell-Boltzmann* speed distribution in Eq. (111) may be also viewed as distribution of sizes of particle clusters, if expressed in dimensionless form it can also be viewed as the sizes of "clusters of numbers" or *Hilbert* "condensations". Therefore, a recent study [123] was focused on exploration of possible connections between the result in Eq. (111) and the theoretical findings of *Montgomery* [124] and *Odlyzko* [125] on analytical number theory that has resulted in what is known as *Montgomery-Odlyzko* [124-125] law

"The distribution of the spacing between successive non-trivial zeros of the Riemann zeta function (suitably normalized) is statistically identical with the distribution of eigenvalue spacing in a GUE operator"

The pair correlation of *Montgomery* [124] was subsequently recognized by *Dyson* to correspond to that between the energy levels of heavy elements [126-127] and thus to the pair correlations between eigenvalues of *Hermitian* matrices [128]. Hence, a connection was established between quantum mechanics on the one hand and quantum chaos [129] on the other hand. However, the exact nature of the connections between these seemingly diverse fields of quantum mechanics, random matrices, and *Riemann* hypothesis [126-127] is yet to be understood.

When the oscillator speeds (cluster sizes) in Eq. (111) are normalized through division by the most probable speed (the most probable cluster size) one arrives at *Normalized Maxwell-Boltzmann* NMB distribution function [123]

$$\rho_j = (8/\pi_\beta)\,[(2/\sqrt{\pi_\beta})x_{j\beta}]^2\, e^{-[(2/\sqrt{\pi_\beta})x_{j\beta}]^2} \qquad (119)$$

The additional division by the "measure" $\sqrt{\pi_\beta}/2$ in Eq. (119) is for coordinate normalization as discussed in Ref. 122 and shown in Fig. 18. Direct comparisons between Eq. (119) and the normalized spacing between the zeros of *Riemann* zeta function and the eigenvalues of GUE calculated by *Odlyzko* [125] are shown in Fig. 17. Therefore, a definite connection has been established between analytic number theory, the kinetic theory of ideal gas, and the normalized spacing between energy levels in quantum mechanics [123].

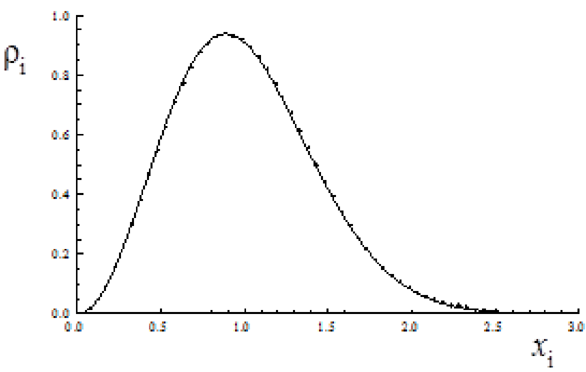


Fig. 17 Probability density of normalized spacing between zeros of *Riemann* zeta function [125] $\gamma_n, 10^{12} \le n \le 10^{12} + 10^5$, normalized spacing between eigenvalues of GUE [125], and the NMB distribution function in Eq. (119).

To further examine the connection between *Riemann hypothesis* and *Maxwell-Boltzmann* kinetic theory of ideal gas the speed distribution is first related to distribution of sizes or wavelengths of number clusters (Fig. 14). According to equations (29)-(30) and (36) particle energy and frequency are related by

$$\varepsilon_{tj} = m v_j^2 = h\nu_j \qquad (120)$$

Therefore, the normalized spacing between energy levels can be expressed in terms of the normalized spacing between frequencies of virtual oscillators as

$$(\varepsilon_{tj} - \varepsilon_{ti})/\varepsilon_{mp} = (\nu_j - \nu_i)/\nu_{mp} \qquad (121)$$

Because of *Boltzmann's* equipartition principle the particles' random rotational and vibrational (pulsational) kinetic energy in two directions $(\theta+, \theta-)$ and $(r+, r-)$ will be equal to their translational kinetic energy in Eq. (120) and follow *Planck* law in Eq. (80). Also, the corresponding momenta of all three degrees of freedom will be randomly distributed and once properly normalized should follow NMB distribution in Eq. (119). Therefore, parallel to Eq. (30) the rotational counterpart of Eq. (120) is expressed as

$$\begin{aligned}\varepsilon_{rj\beta} &= I\langle\omega^2_{j\beta\theta+}\rangle/2 + I\langle\omega^2_{j\beta\theta-}\rangle/2 = I\langle\omega^2_{j\beta\theta+}\rangle \\ &= m < r^2\omega^2_{j\beta\theta+} >^{1/2} < (2\pi r)^2 \nu^2_{j\beta} >^{1/2} \\ &= (m < u^2_{j\beta} >^{1/2} < \lambda^2_{j\beta} >^{1/2}) < \nu^2_{j\beta} >^{1/2} = h\nu_{j\beta} \qquad (122)\end{aligned}$$

where I is the moment of inertia and by equipartition principle $<\omega^2_{\beta\theta+}> = <\omega^2_{\beta\theta-}>$. By Eq. (122) the normalized spacing between rotational energy levels will also be related to the normalized spacing between frequencies of oscillators in Eq. (121).

Following the classical methods [92] for the vibrational degree of freedom the potential energy of harmonic oscillator is expressed as

$$\begin{aligned}\varepsilon_{vj\beta} &= \kappa\langle x^2_{j\beta+}\rangle/2 + \kappa\langle x^2_{j\beta-}\rangle/2 = \kappa\langle x^2_{j\beta+}\rangle \\ &= m\langle\omega^2_{j\beta+}\rangle\langle x^2_{j\beta+}\rangle = m\langle 2\pi x_{j\beta+}\rangle^2\langle\nu^2_{j\beta}\rangle \\ &= (m < u^2_{j\beta} >^{1/2} < \lambda^2_{j\beta} >^{1/2}) < \nu^2_{j\beta} >^{1/2} = h\nu_{j\beta} \qquad (123)\end{aligned}$$

where χ is the spring constant and $\tilde{\omega} = \sqrt{\chi/m}$ (Ref. 92). Similar to equations (120) and (122), by Eq. (123) the normalized spacing between potential energy levels of harmonic oscillator are also related to the normalized spacing between frequencies of virtual oscillators in Eq. (121). In summary, the normalized spacing between energy levels for translational, rotational, and vibrational motions are related to their corresponding normalized frequencies by

$$(\varepsilon_{qj} - \varepsilon_{qi})/\varepsilon_{qmp} = (\nu_{qj} - \nu_{qi})/\nu_{qmp} \quad , \quad q = t, r, v \qquad (124)$$

Therefore, the classical model of diatomic molecule with rigid-body rotation and harmonic vibration [92] is herein considered to also possess an *internal* translational harmonic motion. The internal translational degree of freedom is associated with thermodynamic pressure and may therefore be called internal potential energy. In addition to this internal harmonic translation there will be an external harmonic motion due to the peculiar translational velocity in Eq. (3) and a corresponding external potential energy (Fig. 15). In Sec. 11, it will be shown that the external potential energy appears in *Schrödinger* equation (206) and acts as *Poincaré* stress [117-119] that is responsible for "particle" stability. The particle trajectory under all four degrees of freedom namely the three internal translational, rotational and vibrational motions and the external peculiar motion will be quite complicated. Clearly, addition of radial oscillations about the center of mass to the rigid rotator will result in particle motion on a radial wave, *de Broglie* wave, along the circumference of the otherwise circular particle trajectory.

Now that the normalized spacing between energy levels have been related to the normalized spacing between oscillator frequencies ν_j in Eq. (124), the latter should be connected to the zeros of *Riemann* zeta function. The zeros of *Riemann* zeta function are related to prime numbers through *Euler*'s Golden Key [126]

$$\zeta(s) = \sum_n \frac{1}{n^s} = \prod_{p_j} (1 - p_j^{-s})^{-1} \tag{125}$$

where s = a +ib is a complex number. Clearly, the zeros of zeta function in Eq. (125) will coincide with the zeros of the powers of primes

$$p_j^s = p_j^{a+ib} = 0 \tag{126}$$

It is most interesting that according to his *Nachlass* [127] *Riemann* was working on the problem of *Riemann Hypothesis* and the hydrodynamic problem of stability of rotating liquid droplets simultaneously. In view of Eq. (119) and the connections between normalized energy and frequency spacing shown in Fig. 17, it is natural to consider the situation when particle frequencies are given by integral powers of prime numbers as

$$\nu_j = p_j^1, p_j^2, p_j^3 = \ldots = p_j^N = \square_{p_j} \tag{127}$$

in harmony with *Gauss's* clock calculator [127] and *Hensel's* $p_{j\beta}$-adic numbers $\square_{p_j}$ (Ref. 126).

Now, since the atomic energy of particles must be quantized according to *Planck* formula $\varepsilon = h\nu$ by Eq. (127) one writes

$$mv_j^2 = h\nu_j = hp_j^N = h\square_{p_j} \tag{128}$$

that by equations (30)-(31) and (36) suggest that the particle velocity may be expressed as

$$\mathbf{v}_{j\pm} = \sqrt{h/m}\, p_j^{N/2} e^{i\theta} = \sqrt{h/m}\, p_j^{N/2 + iNb\ln p_j} = \sqrt{h/m}\, \square_{p_j}^{1/2+ib} \tag{129}$$

where the angle $\theta = Nb\ln p_j$ corresponds to the direction of velocity vector. Therefore, by Eq. (129) the particle energy consistent with Eq. (128) becomes

$$\varepsilon_j = m(\mathbf{v}_j.\mathbf{v}_j^*) = hp_j^N = h\square_{p_j} \tag{130}$$

Also, the dependence of particle speed on $\square_{p_j}$ will be obtained from equations (128) and (129) as

$$v_{j\pm} = \sqrt{h/m}\, p_j^{N/2} = \sqrt{h/m}\, \square_{p_j}^{1/2} \tag{131}$$

that by the relation $v_j = \lambda_j \nu_j$ gives particle wavelength

$$\lambda_{j\pm} = \sqrt{h/m}\, \square_{p_j}^{-1/2} \tag{132}$$

In the following it is shown that with the value a = ½ as the position of the critical line in accordance with *Riemann Hypothesis* [126, 127] the zeros of particle velocities from Eq. (129) will coincide with the zeros of *Riemann* zeta function in Eq. (126). One also expects the zeros of particle velocity from Eq. (129) to coincide with the zeros of particle speed in Eq. (131), and those of particle energy and hence frequency in Eq. (128). Thus, at thermodynamic equilibrium all points of the normalized *Maxwell-Boltzmann* distribution in Fig. 17 correspond to stochastically stationary states of clusters of particles undergoing random translational, rotational, vibrational motions while satisfying the principle of detailed balance of quantum mechanics through continuous transitions between clusters (energy levels) further discussed in Sec. 11.

At thermodynamic equilibrium *Maxwell-Boltzmann* speed distribution in Eq. (119) corresponds to translational, rotational, and vibrational particle velocities that must follow *Gaussian* distribution like Fig. 5. Also, space isotropy requires that the translational, rotational, and vibrational momenta of particles in two directions (x_+, x_-), (θ_+, θ_-), (r_+, r_-) be equal in magnitude and opposite in direction such that by Eq. (129) at the zeros of particle velocity

$$\mathbf{v}_{j\pm}^{1/N} = \sqrt[2N]{h/m}\, p_j^{1/2} e^{\pm i\ln(p_j^b)} = \sqrt[2N]{h/m}\, p_j^{1/2}[\cos(\ln p_j^b) \pm i\sin(\ln p_j^b)] = 0 \tag{133a}$$

the corresponding particle momenta become identically zero. Therefore, one expects "stationary states" at mean translational position $(\bar{x} = 0)$, mean angular position $(\bar{\theta} = 0)$, and mean radial position $(\bar{r} = 0)$ at which particle velocities (v_t, v_θ, v_v) and hence energies $(\varepsilon_t, \varepsilon_\theta, \varepsilon_v)$ vanish. Hence, another argument for the position of the critical line being at a = ½ could be based on symmetry requirements for passage across stationary states and the *Gaussian* velocity distribution (Fig. 5) at equilibrium. Because by Eq. (124) all three forms of energy could be expressed as products of *Planck* constant and frequency, one identifies the zeros of velocities in Eq. (129) as the "stationary states" of particle's translational, rotational, and vibrational momenta. The connection between the kinetic theory of ideal gas and *Montgomery-Odlyzko* law shown in Fig. 17 involves spacing between particle speeds in Eq. (131) whereas the zeros of *Riemann* zeta function are related to stationary states or the zeros of particle velocities in Eq. (129). However, the zeroes of particle speeds must coincide with the zeros of particle velocities that from Eq. (133a) are given by

$$p_j^{1/2+ib} = 0 \qquad (133b)$$

Comparison between the zeros of particle velocity at stationary states in Eq. (133b) and the zeros of *Riemann* zeta function in Eq. (126) shows that they will coincide if a = ½ namely on the critical line in accordance with *Riemann* hypothesis.

It is interesting to examine the connection between analytical number theory and *Riemann* hypothesis in terms of particle wavelengths in Eq. (132) hence quantized spatial coordinates rather than particle speeds in Eq. (131). In a recent study [122] a logarithmic system of coordinates was introduced as

$$x'_\beta = \ln N_{A\beta} \qquad (134)$$

whereby the spatial distance of each statistical field (Fig. 1) is measured on the basis of the number of "atoms" of that particular statistical field $N_{A\beta}$. With definition in Eq. (134) the counting of numbers must begin with the number zero naturally since it corresponds to a single atom. The number of atoms in the system is expressed as [122]

$$N_{AS\beta} = \left(N_{AE\beta}\right)^{N_{ES\beta}} \qquad (135)$$

where $(N_{AS\beta}, N_{ES\beta}, N_{AE\beta})$ respectively refer to the number of atoms in the system, number of elements in the system, and number of atoms in the element. The hierarchy of the resulting normalized coordinates is shown below [122]

$$\begin{array}{l} \cdots \\ L_{\infty\beta} \text{———————} 1_\beta \text{———} 0_\beta \\ \qquad\qquad\qquad L_{\infty\beta-1} \text{——} 1_{\beta-1} \text{—} 0_{\beta-1} \\ \qquad\qquad\qquad\qquad\qquad\qquad \cdots \end{array} \qquad (136)$$

The exact connections between spatial coordinates of hierarchies of statistical fields (Fig. 1) will involve the important concept of re-normalization [130,131]. Normalization in Eq. (136) is based on the concept of "dimensionless" or "measureless" numbers [122]

$$x_\beta = x'_\beta / \delta_\beta = N_{ES\beta} \qquad (137)$$

where the "measure" δ_β is defined by *Gauss's* error function as

$$\delta_\beta = \int_{0_{\beta-1}}^{\infty_{\beta-1}} e^{-x_{\beta-1}^2} dx_{\beta-1} = \sqrt{\pi_{\beta-1}} / 2 \qquad (138)$$

In view of equations (137)-(138), the range $(-1_\beta, 1_\beta)$ of the outer coordinate x_β will correspond to the range $(-\infty_{\beta-1}, \infty_{\beta-1})$ of the inner coordinate $x'_{\beta-1}$ leading to the coordinate hierarchy schematically shown in Fig. 18.

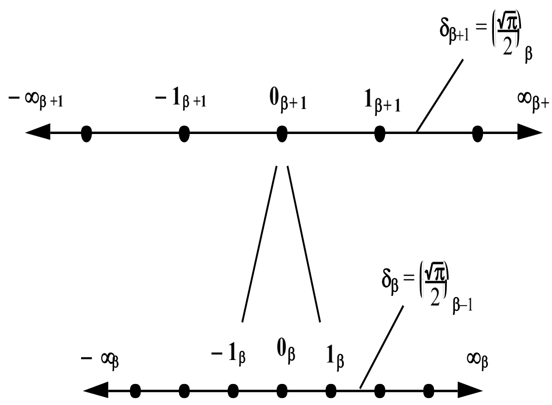


Fig. 18 Hierarchy of normalized coordinates associated with embedded statistical fields [122].

As discussed above and in [123] one naturally considers the prime numbers $p_{1\beta} = 2$, $p_{2\beta} = 3$, $p_{3\beta} = 5$, …, to be the "atoms" of the statistical field at scale β that may also be viewed as different atomic "species". However in view of equations (127) and (132) space quantization will be based on the inverse power of $p_{j\beta}$-adic numbers such that by Eq. (135)

$$N_{ASj\beta}^{1/2} = \left(N_{AEj\beta}\right)^{N_{ESj\beta}/2} = \left(p_{j\beta}\right)^{-N_{ESj\beta}/2} \qquad (139)$$

that parallel to Eq. (132) is expressed as

$$\lambda_{j\beta} = \sqrt{h/m\, p_{j\beta}^{-N_{ESj\beta}/2}} = \sqrt{h/m\, \square_{mpj\beta}^{-1/2}} \qquad (140)$$

The reason for the choice of primes is that they represent indivisible "atoms" of arithmetic out of which all natural numbers at any scale could be constructed. The quantized wavelengths in Eq. (140) like Eq. (132) will correspond to quantized frequencies in Eq. (127), energies in Eq. (130), and speeds in Eq. (131) all involving $p_{j\beta}$-adic numbers $\square_{p_\beta}$ that were employed in the construction of *Adele* space of noncommutative spectral geometry of *Connes* [126, 127, 132].

The wavelength in Eq. (140) is next normalized with respect to the most probable cluster size

$$\lambda_{mj\beta} = \sqrt{h/m\, p_{j\beta}^{-\bar{N}/2}} = \sqrt{h/m\, \square_{mpj\beta}^{-1/2}} \qquad (141)$$

and since at thermodynamic equilibrium the entire distribution function (Fig. 17) is stochastically stationary at all particle speeds $v_{j\beta} = \lambda_{j\beta} \nu_{j\beta} = \text{constant}$ equations (102), (103), (140), and (141) give

$$x_{j\beta}^{-1} = \frac{v_{mp\beta}}{v_{j\beta}} = \frac{\lambda_{j\beta}}{\lambda_{m_{j\beta}}} = \frac{p_{\beta j}^{-N_j/2}}{p_{\beta j}^{-\bar{N}/2}} = \frac{\square_{p_{j\beta}}^{-1/2}}{\square_{mp_{j\beta}}^{-1/2}} \tag{142}$$

With $p_{j\beta}$-adic numbers $\square_{p_\beta}$ incorporated into the quantized wavelengths in Eq. (142) one arrives at dimensionless speeds in Eq. (131) and hence the *Normalized Maxwell-Boltzmann* (NMB) distribution in Eq. (119) for the prime "specie" $p_{j\beta}$.

To summarize, first the normalized spacing between energy levels were related to the normalized spacing between oscillator frequencies in Eq. (124). Next, oscillator frequencies were taken as $p_{j\beta}$-adic numbers in Eq. (127) and the spacing between frequencies and zeros of energy levels in Eq. (128) were related to the spacing between "stationary states" or zeros of particle velocity in Eq. (129), that were in turn related to the zeros of *Riemann* zeta function by Eq. (126) through *Euler's* golden key in Eq. (125). The model therefore provides a physical explanation of *Montgomery-Odlyzko* [126, 127] law as well as *Hilbert-Polya* [126] conjecture that spacing between the zeros of *Riemann* zeta function are related to spacing between eigenvalues of *Hermitiam* matrices. The close agreement shown in Fig. 17 is because at equilibrium by Eq. (124) the distribution of normalized particle energy in Eq. (130) and speed in Eq. (131) are related by *Maxwell-Boltzmann* distribution in Eq. (119). Moreover, because particle energy vanishes at the stationary states given by Eq. (133b) the energy spectrum will correspond to *absorption spectrum* in accordance with the prediction of noncommutative spectral geometry of *Connes* [132]. For complete resolution of *Riemann Hypothesis* one must now identify the appropriate normalization method to relate the zeros of *Riemann* zeta function by Eq. (125) hence Eq. (126) to the zeros of particle velocity in Eq. (129) hence Eq. (133b) that is simpler than the *Riemann-Siegel* formula [126].

Since each partial density in Eq. (119) corresponds to single "prime" specie, one next constructs a mixture density by summing all the partial probability densities of all "prime" species

$$\rho_\beta \,\square\, \sum_j \rho_{j\beta} \tag{143}$$

to arrive at

$$\rho_\beta = (8/\pi_\beta)\,[(2/\sqrt{\pi_\beta})x_\beta]^2\, e^{-[(2/\sqrt{\pi_\beta})x_\beta]^2} \tag{144}$$

where

$$\sum x_{j\beta}^2 = \sum \frac{v_{j\beta}^2}{v_{m\beta}^2} = \sum \frac{m_j v_{j\beta}^2}{m v_{m\beta}^2} = \frac{1}{v_{m\beta}^2} \sum Y_j v_{j\beta}^2 .$$

$$= \frac{v_\beta^2}{v_{m\beta}^2} = x_\beta^2 \tag{145}$$

because the mean energy of all species are identical at thermodynamic equilibrium $m_j v_{mj\beta}^2 = m_i v_{mi\beta}^2 = kT$. The grand ensemble of NMB $p_{j\beta}$-adic statistical fields in Eq. (144) will have a corresponding GUE that could be identified as *Connes* [124-127, 132] *Adele* space $\mathfrak{A}_\beta$ at a particular scale β. Therefore, the normalized *Adele* space $\mathfrak{A}_\beta$ at any particular scale β is constructed from superposition of infinite NMB distribution functions like Fig. 17 corresponding to atomic specie $p_{j\beta}$ and cluster sizes in the form of $p_{j\beta}$-adic numbers.

With prime numbers $p_{j\beta}$ as atomic species the spacing between wavelengths of number clusters or *Hilbert* condensations [126] are related to the spacing between particle speeds by Eq. (142) that are in turn related to the normalized spacing between energy levels by Eq. (124). The connection to quantum mechanics is further evidenced by direct derivation of *Maxwell-Boltzmann* speed distribution in Eq. (111) from *Planck* distribution in Eq. (80) discussed in Sec. 7. Also, because the GUE associated with Eq. (144) is based on $p_{j\beta}$-adic type numbers, the normalized spacing between its eigenvalues should be related to the normalized spacing between the zeros of *Riemann* zeta function according to the theory of noncommutative geometry of *Connes* [132]. Although the exact connection between noncommutative geometry and *Riemann* hypothesis is yet to be understood according to *Connes* [127]

"The process of verification can be very painful: one's terribly afraid of being wrong…it involves the most anxiety, for one never knows if one's intuition is right- a bit as in dreams, where intuition very often proves mistaken"

the model suggested above may help in the clarification of the physical foundation of such a mathematical theory.

The scale invariance of the model, possible electromagnetic nature of all matter discussed in Sec. VI, as well as the connection between analytic number theory and the kinetic theory of ideal gas (Fig. 17) all appear to confirm the perceptions of *Sommerfeld* [133]

"Our spectral series, dominated as they are by integral quantum numbers, correspond, in a sense, to the ancient triad of the lyre, from which the Pythagoreans 2500 years ago inferred the harmony of the natural phenomena; and our quanta remind us of the role which the Pythagorean doctrine seems to have ascribed to the integers, not merely as attributes, but as the real essence of physical phenomena."

as well as those of *Weyl* [134].

Since the *physical space* or *Casimir* [73] vacuum itself is identified as a *fluid* and described by a statistical field [34, 83], it will have a spectrum of energy levels given by *Schrödinger* equation (206) that in view of *Heisenberg* [135, 34, 83] matrix mechanics will be described by noncommutative spectral geometry of *Connes* [132]. Hence, the above results in harmony with the perceptions of *Pythagoras* and *Plato* suggest that pure "numbers" maybe the basis of all that is physically "real" [34, 136].

IX. Invariant Transport Coefficients and Hierarchies of Absolute Zero Temperatures and Vacua

Following *Maxwell* [36, 39] the scale invariant definition of kinematic viscosity $\kappa_\beta = \eta_\beta / \rho_\beta$ may be expressed as

$$\kappa_\beta = \frac{1}{3}\ell_\beta u_\beta = \frac{1}{3}\lambda_{\beta-1} v_{\beta-1} \tag{146}$$

At the scale of β = e corresponding to equilibrium eddy dynamics EED (Fig. 1) equation (146) gives *Boussinesq* eddy diffusivity [137]

$$\kappa_e = \frac{1}{3}\ell_e u_e = \frac{1}{3}\lambda_c v_c \tag{147}$$

On the other hand, the kinematic viscosity at LCD scale involves the "atomic" length ℓ_c or the molecular mean free path λ_m that appears in *Maxwell's* formula for kinematic viscosity [30]

$$\kappa_c = \frac{1}{3}\ell_c u_c = \frac{1}{3}\lambda_m v_m \tag{148}$$

associated with viscous dissipation in fluid mechanics. It is important to note that the model predicts a finite kinematic viscosity for the scale β = s i.e. electrodynamics scale when energy can be dissipated to *Casimir* [73] vacuum at the lower chromodynamics scale

$$\kappa_s = \frac{1}{3}\ell_s u_s = \frac{1}{3(2\pi)}\lambda_k v_k = \frac{1}{3(2\pi)}\frac{\lambda_k v_k m_k}{m_k}$$

$$= \frac{1}{3(2\pi)}\frac{h_k}{m_k} = \frac{\hbar}{3m_k} \tag{149}$$

in exact agreement with the result of *de Broglie* [3] provided that the equality of particle mean free path and wavelength $\lambda_k = \lambda_k$ holds. Therefore, *Ohmic* dissipation could occur by transfer of energy from electrons into photon gas constituting *Casimir* [73] vacuum. In view of Fig. 1, it is natural to expect that energy of photon field at chromodynamics scale could be dissipated into a sub-photonic tachyon field that constitutes a new vacuum called vacuum-vacuum. According to the model electromagnetic waves propagate as viscous flow due to "viscosity" of radiation [41] or gravitational viscosity [138] in harmony with the well-known concept of "*tired light*".

In view of the definition of *Boltzmann* constant in equations (30b), (33) and (36), *Kelvin* absolute temperature is related to the most probable wavelength of oscillations by Eq. (90) and hence becomes a length scale. Therefore, *Kelvin* absolute temperature approaches zero through an infinite hierarchy of limits as suggested by Eq. (136) and Fig. 18. In other words, one arrives at a hierarchy of "absolute zero" temperatures defined as

$$\begin{array}{l} \quad\quad\quad\quad\quad\quad\quad\quad \cdots \\ \quad\quad T_\beta = 0_\beta = T_{\beta-1} = 1_{\beta-1} \\ T_{\beta-1} = 0_{\beta-1} = T_{\beta-2} = 1_{\beta-2} \\ \cdots \end{array} \tag{150}$$

As discussed in Sec.VI, *physical space* could be identified as a tachyonic fluid [139] that is the stochastic ether of *Dirac* [111] or "hidden thermostat" of *de Broglie* [3]. The importance of *Aristotle's* ether to the theory of electrons was emphasized by *Lorentz* [140-141]

"I cannot but regard the ether, which is the seat of an electromagnetic field with its energy and its vibrations, as endowed with certain degree of substantiality, however different it may be from all ordinary matter"

Also, in Sec. 6 photons were suggested to be composed of a large number of much smaller particles [139], like neutrinos that themselves are composed of large numbers of *tachyons* [107]. Therefore, following *Casimir* [73] and in harmony with Eq. (150) one expects a hierarchy of vacua

$$(\text{vacuum-vacuum})_\beta = (\text{vacuum})_{\beta-1} \tag{151}$$

as one attempts to resolve the granular structure of physical space and time at ever smaller scales described by the coordinates in Eq. (136).

The hierarchies of coordinates in Eq. (136) and vacua in Eq. (151) will have an impact on the important recurrence theory of *Poincaré* [44] and its implication to *Boltzmann's* expression of thermodynamic entropy

$$S = k \ln W \tag{152}$$

In particular, the conflict between *Poincaré* recurrence theory and thermodynamic irreversibility emphasized by *Zermelo* [142] should be reexamined. As was emphasized by *Boltzmann* [143] *Poincaré* [44] recurrence theory cannot be applied to thermodynamic systems because such systems cannot be truly isolated. That is, the unattainable nature of absolute vacuum-vacuum in Eq. (151) makes isolation of all physical systems impossible. The same limitation will apply to the entire universe when our universe is considered to be just one universe as an open system among others according to *Everett's* many universe theory described by *DeWitt* [144]. The hierarchy of vacua in Eq. (151) is in harmony with the inflationary theories of cosmology [145-148] and the finite pressure of *Casimir* [73] vacuum given by the modified *van der Waals* law of corresponding states [149]

$$p_r = \frac{1}{Z_c}\left[\frac{T_r}{v_r - 1/3} - \frac{9}{8v_r^2} + Z_c - \frac{3}{8}\right] \tag{153}$$

where Z_c is the critical compressibility factor.

Clearly, the nature of thermodynamic irreversibility will be impacted by both the hierarchical definition of time discussed in [34] as well as the cascade of embedded statistical fields shown in Fig. 1. For example, let us consider at EED scale a hydrodynamic system composed of 10^2 fluid elements each of which composed of 10^2 clusters of eddies. Next, let eddy, defined as the most probable size ensemble of molecular-clusters at ECD scale, be composed of 10^2 mean molecular-clusters each containing 10^8 molecules. Let us next assume that only the cluster i of the eddy j of the fluid element f contains molecules of type B and that all other clusters of all other eddies are composed of molecules of type A. The system is then allowed to fully mix at the molecular level. The thermodynamic reversibility will now require that 10^8 type B molecules to first become unmixed at hydrodynamic scale by leaving (10^2-1) fluid elements and collecting in the fluid element f. Next, all 10^8 type B molecules must leave (10^2-1) eddies and collect in the eddy e_{jf}. Finally 10^8 type B molecules must leave (10^2-1) clusters of eddy j and collect in the cluster c_{ijf}. Clearly the probability of such preferential motions to lead to immixing against all possible random motions will be exceedingly small. When the above hierarchy of mixing process is extended to yet smaller scales of molecular-dynamics, electrodynamics, and chromo-dynamics, the probability for reversibility becomes almost zero bordering impossibility in harmony with perceptions of *Boltzmann* [38]. The broader implications of the hierarchy of coordinate limits in Eq. (136) to the internal set theory of *Nelson* [150] and the recurrence theory of *Poincaré* [44] require further future investigations.

X. Invariant Forms of the First Law of Thermodynamics and Definition of Entropy

In this section, *Boltzmann* statistical mechanics for ideal monatomic gas discussed in Sec. 7 will be applied to arrive at invariant forms of *Boltzmann* equation for entropy and the first law of thermodynamics. The results also suggest a new perspective of the nature of entropy by relating it to more fundamental microscopic parameters of the thermodynamic system.

As stated in Sec. 3, when the three velocities $(\mathbf{u}_m, \mathbf{v}_m, \mathbf{V}'_m)$ in Eq. (3) are all random the system is composed of an ensemble of molecular clusters and single molecules under equilibrium state and one obtains from Eq. (3)

$$m<\mathbf{u}^2_{x+\beta}>=m<\mathbf{v}^2_{x+\beta}>+m<\mathbf{V}'^2_{x+\beta}>$$

$$\hat{\varepsilon}_t=\hat{\varepsilon}_{tke}+\hat{\varepsilon}_{pe}=\hat{\varepsilon}_{tke}+p\hat{v} \qquad (154)$$

as the sum of the internal translational kinetic and "external" potential energies since $<2\mathbf{v}_{x+\beta}\mathbf{V}'_{x+\beta}>=0$. One next allows the monatomic particles to also possess both rotational $\hat{\varepsilon}_{rke}$ and vibrational (pulsational) $\hat{\varepsilon}_{vke}$ kinetic energy [31] and invokes *Boltzmann*'s principle of equipartition of energy such that

$$m<\mathbf{v}^2_{t\beta x+}> \;=\hat{\varepsilon}_{tke}=\hat{\varepsilon}_{rke}=\hat{\varepsilon}_{vke}=\hat{\varepsilon}_{pe} \qquad (155)$$

With the results in Eqs. (154)-(155) the total energy of the particle could be expressed as [32]

$$\hat{h}_\beta=\hat{u}_\beta+p_\beta\hat{v} \qquad (156)$$

where $\hat{h}_\beta$ is the enthalpy, $\hat{v}_\beta$ is the volume and

$$\hat{u}_\beta=\hat{\varepsilon}_{tke}+\hat{\varepsilon}_{rke}+\hat{\varepsilon}_{ke}=3\hat{\varepsilon}_{tke} \qquad (157)$$

is the internal energy per molecule such that by Eq. (156)

$$H_\beta=U_\beta+p_\beta V_\beta \qquad (158)$$

where $V_\beta=N_\beta\hat{v}_\beta$, $H_\beta=N_\beta\hat{h}_\beta$, and $U_\beta=N_\beta\hat{u}_\beta$ are respectively volume, enthalpy, and internal energy.

In accordance with the perceptions of *Helmholtz*, one may view Eq. (158) as the first law of thermodynamics

$$Q_\beta=U_\beta+W_\beta \qquad (159)$$

when reversible heat and work are defined as [31]

$$Q_\beta=H_\beta=T_\beta S_\beta \qquad (160)$$

and

$$W_\beta=p_\beta V_\beta \qquad (161)$$

and S_β is the entropy. For an ideal gas equations (159) and (160) lead to

$$Q_\beta=T_\beta S_\beta=U_\beta+p_\beta V_\beta=H_\beta$$

$$=3N_\beta kT_\beta+N_\beta kT_\beta=4N_\beta kT_\beta \qquad (162)$$

When photons are considered as monatomic ideal gas integration of Eq. (81) and maximization of Eq. (58) are known to lead to internal energy and entropy [92-94]

$$U_\beta=\frac{8\pi V_\beta(kT)^4}{15(hc)^3} \quad , \quad S_\beta=\frac{32\pi^5V_\beta k^4T^3}{45(hc)^3} \qquad (163)$$

Since *Poisson* coefficient of photon is $\gamma=\hat{c}_p/\hat{c}_v=4/3$ the result in Eq. (162) leads to

$$T_\beta S_\beta=\frac{4}{3}U_\beta \qquad (164)$$

in accordance with Eq. (163). Also, since for an ideal gas equations (157) and (159) give

$$U_\beta=3\varepsilon_{tke}=3p_\beta V=3N_\beta kT_\beta \qquad (165)$$

by Eq. (162) entropy per photon $\hat{s}_k=S/N$ becomes [32]

$$\hat{s}_k = 4k \tag{166}$$

as compared to

$$\hat{s}_k = 3.6k \tag{167}$$

based on the classical model [94].

The discrepancy between equations (166) and (167) is due to the classical formulation [94] for the number of photons in a given volume V_β. It is possible to express the total potential energy from the integration of mean energy and number density of oscillators over for spherical number density space as

$$\int_0^N <\varepsilon> dN = \int_0^N kTdN = NkT$$

$$= \int_0^N \bar{n}\varepsilon(4\pi N_x^2)dN = \int_0^N \frac{h\nu(8\pi N_{x+}^2)}{e^{h\nu/kT}-1}dN$$

$$= \int_0^N \frac{h\nu(8\pi N^2/3)}{e^{h\nu/kT}-1}dN \tag{168}$$

One next considers the relation between the number of quantized oscillators in a cube of size L = L_x = L_y = L_z as

$$N_{x+} = L/\lambda = L\nu/c \tag{169}$$

and the isotropy condition

$$N^2 = N_{x+}^2 + N_{y+}^2 + N_{z+}^2 = 3N_{x+}^2 \tag{170}$$

Because *de Broglie* [2] "matter wave packets" or *Heisenberg-Kramers* [72] virtual oscillators are now considered to represent actual particles of ideal gas according to Eq. (169) one requires integral numbers N of the full wavelength λ, as opposed to the conventional half wavelength λ/2 to fit within the cavity of length L. Substituting from equations (169)-(170) into Eq. (168) gives

$$NkT = \frac{8\pi h}{3}\int_0^\infty \frac{\nu(L/c)^2\nu^2}{e^{h\nu/kT}-1}(L/c)d\nu =$$

$$\frac{8\pi hL^3}{3c^3}\int_0^\infty \frac{\nu^3}{e^{h\nu/kT}-1}d\nu = \frac{8\pi V}{3}\frac{(kT)^4}{(hc)^3}\int_0^\infty \frac{x^3}{e^x-1}dx$$

$$= \frac{8\pi^5 V}{45}\frac{(kT)^4}{(hc)^3} \tag{171}$$

Hence, the number of photons in volume V is given by Eq. (171) as

$$N = \frac{8\pi^5 V}{45}\left(\frac{kT}{hc}\right)^3 \tag{172}$$

as compared to the classical result [94]

$$N = 8\pi V\left(\frac{kT}{hc}\right)^3 \times 2.404 \tag{173}$$

Also, from equations (172) and (162) one obtains the internal energy [32]

$$U = 3NkT \tag{174}$$

that leads to the expected specific heat at constant volume

$$\tilde{c}_v = 3R^o \tag{175}$$

that is in accordance with Eq. (106)

The result in Eq. (174) only approximately agrees with the classical expression for the internal energy [93-94]

$$U = \frac{\pi^4}{30}\frac{NkT}{\zeta(3)} \tag{176}$$

that leads to

$$U = 2.701NkT \tag{177}$$

According to Eq. (171) the expression in Eq. (176) should involve zeta function of 4 rather than 3 such that

$$U = \frac{\pi^4}{30}\frac{NkT}{\zeta(4)} = 3NkT \tag{178}$$

that agrees with Eq. (174) and by Eq. (162) leads to the ideal gas law [32]

$$pV = NkT \tag{179}$$

instead of the classical result [93]

$$pV = 0.900NkT \tag{180}$$

for equilibrium radiation.

Concerning the results in equations (177) and (180) it was stated by *Yourgrau et al.* [93]

"The reader will not fail to recognize the close resemblance between relations and their counterparts pertaining to a classical ideal gas"

With the modified results in equations (174) and (179) the correspondence between photon gas and classical ideal gas becomes exact thus closing the gap between radiation and gas theory discussed by *Darrigol* [151] and *Stachel* [152]. The important relation between radiation pressure and internal energy [92-94]

$$p = U/3V = \hat{u}/3\hat{v} = u/3 \quad (181)$$

is exactly satisfied by equations (174) and (179) and closely but only approximately satisfied by equations (173) and (180).

Since *Kelvin* absolute temperature scale is identified as a length scale by Eq. (90), thermodynamic temperature relates to spatial and hence temporal "measures" of the physical space [34] or *Casimir* [73] vacuum. Therefore, the hierarchy of limiting zero temperatures in Eq. (150) will be related to hierarchy of "measures" that are employed to renormalize [130, 131] "numbers" by equations (137)-(138) and arrive at "dimensionless" coordinates (Fig. 18) as discussed earlier [122]. For an ideal monatomic gas one has the $\tilde{c}_p = 4R^o$ and $\tilde{c}_v = 3R^o$ such that Eq. (162) reduces to the identity

$$4_\beta = 3_\beta + 1_\beta \quad (182)$$

Therefore, the mathematical relation (182) always holds for statistical fields of any scale (Fig. 1) as the thermodynamic temperature in Eq. (150) approaches the "absolute zero" $T_\beta \to 0_\beta$ ($T_{\beta-1} \to 1_{\beta-1}$) associated with coordinates of that particular scale in Eq. (136).

The result in Eq. (162) could be also obtained from the *Gibbs* equation

$$T_\beta dS_\beta = dU_\beta + p_\beta dV_\beta - \sum_j \hat{\mu}_{j\beta} dN_j \quad (183)$$

, *Gibbs-Duhem* equation

$$S_\beta dT_\beta = V_\beta dp_\beta - \sum_j N_j d\hat{\mu}_{j\beta} \quad (184)$$

and the equality of molar *Gibbs* free energy and chemical potential $\hat{g}_j = \hat{\mu}_j$ of j specie. Addition of equations (183) and (184) leads to *Euler* equation

$$d(T_\beta S_\beta) = dU_\beta + d(p_\beta V_\beta) - d\sum_j \hat{g}_{j\beta} N_{j\beta} = dU_\beta + d(p_\beta V_\beta) - dG_\beta \quad (185)$$

At the state of thermodynamic equilibrium $dG_\beta = 0$ equation (185) leads to Eq. (162) upon integration and application of *Nernst-Planck* third law of thermodynamics requiring $\tilde{h}_\beta \to 0$ in the limit $T_\beta \to 0$ (Ref. 31).

Next, the invariant *Boltzmann* equation (152) for entropy is introduced as

$$S_{j\beta} = k \ln W_{j\beta} \quad (186)$$

Hence

$$\sum_j S_{j\beta} = k \sum_j \ln W_{j\beta} = k \ln \prod_j W_{j\beta} \quad (187)$$

or

$$S_\beta = k \ln W_\beta \quad (188)$$

Substituting in Eq. (186) from Eq. (60) and applying *Stirling's* formula gives

$$S_{j\beta} = k \ln W_{j\beta} = k \ln(g_{j\beta}^{N_{j\beta}} / N_{j\beta}!) = k[N_{j\beta} \ln g_{j\beta} - N_{j\beta} \ln N_{j\beta} + N_{j\beta}] = kN_{j\beta}[\ln(g_{j\beta}/N_{j\beta}) + 1] \quad (189)$$

Also, substituting for *Boltzmann* distribution from Eq. (67) into Eq. (189) leads to

$$S_{j\beta} = kN_{j\beta}[\varepsilon_{j\beta}/kT - \alpha\hat{\mu}_{j\beta}/kT + 1] \quad (190)$$

or

$$TS_{j\beta} = N_{j\beta}\varepsilon_{j\beta} + N_{j\beta}kT - \alpha\hat{\mu}_{j\beta}N_{j\beta} = U_{j\beta} + p_{j\beta}V_\beta - \alpha\hat{g}_{j\beta}N_{j\beta} = H_{j\beta} - \alpha G_{j\beta} \quad (191)$$

Since *Gibbs* free energy is by definition

$$G_{j\beta} = H_{j\beta} - TS_{j\beta} \quad (192)$$

one obtains from equations (191) and (185)

$$\alpha = 1 \quad (193)$$

The results in equations (191) and (193) lead to *Euler* equation

$$TS_{j\beta} = U_{j\beta} + p_{j\beta}V - G_{j\beta} \quad (194)$$

Summation of Eq. (194) over all energy levels results in

$$TS_\beta = U_\beta + p_\beta V - G_\beta \quad (195)$$

Since the principle of equipartition of energy of *Boltzmann* by equations (155) and (162) leads to [32]

$$H_{j\beta} = Q_{j\beta} = TS_{j\beta} = 4N_{j\beta}k_{j\beta}T = 4N_{j\beta}\varepsilon_{jtke} \quad (196)$$

in view of Eq. (30b) the definition of entropy is introduced as *Boltzmann factor*

$$\hat{s}_{j\beta} = 4k_{j\beta} = 4m_\beta \langle u_{j\beta}^2 \rangle^{1/2} \langle v_{j\beta}^2 \rangle^{1/2} \quad (197)$$

that by Eq. (36) when multiplied by "temperature" $T_j = \langle \lambda_{j\beta}^2 \rangle^{1/2}$ gives the "atomic" enthalpy of the energy level j at equilibrium [32]

$$\hat{h}_{j\beta} = 4k_{j\beta}T_{j\beta} = 4kT_{j\beta} = \hat{s}_{j\beta}T_{j\beta} = \hat{s}_{j\beta}T_\beta \quad (198)$$

At EKD scale β = k, the results in equations (196)-(198) are in accordance with Eq. (166) for photon gas. Also, Eq. (198)

is parallel to the way the universal *Boltzmann* constant k times equilibrium temperature T relates to the most probable atomic internal energy

$$\hat{u}_{mp\beta} = 3kT_\beta = m_\beta < u^2_{mp\beta+} > \tag{199}$$

At thermodynamic equilibrium the temperatures of all energy levels are identical $T_{j\beta} = T_\beta$ such that both *Planck* energy spectrum (Fig. 6) and *Maxwell-Boltzmann* speed spectrum (Fig. 16) correspond to an isothermal statistical field at a given thermodynamic temperature T_β. Hence, in Eq. (199) the most probable atomic internal energy is given as the product of temperature and the universal *Boltzmann* constant k from Eq. (36). In Eq. (198) the atomic enthalpy is given as product of temperature and *atomic entropy* defined in Eq. (197) for the energy level j.

The invariant first law of thermodynamics in Eq. (162) when expressed per molecule, per unit mole, and per unit mass appears as [32]

$$\hat{h}_\beta = \hat{u}_\beta + \hat{R}T_\beta = \hat{u}_\beta + kT_\beta = \hat{u}_\beta + p_\beta \hat{v}_\beta \tag{200a}$$

$$\tilde{h}_\beta = \tilde{u}_\beta + \tilde{R}T_\beta = \tilde{u}_\beta + R^o T_\beta = \tilde{u}_\beta + p_\beta \tilde{v}_\beta \tag{200b}$$

$$\bar{h}_{j\beta} = \bar{u}_{j\beta} + \bar{R}_j T_\beta = \bar{u}_{j\beta} + R_j T_\beta = \bar{u}_{j\beta} + p_{j\beta} \bar{v}_\beta \tag{200c}$$

Thus one may express the universal gas constant per molecule, per mole, and per unit mass $(\hat{R}, \tilde{R}, \bar{R}_j)$ as

$$\hat{R} = R^o / N^o = k \tag{201a}$$

$$\tilde{R} = R^o = kN^o \tag{201b}$$

$$\bar{R}_j = R^o / \tilde{W}_j = R^o / (N^o m_j) = k / m_j \tag{201c}$$

XI. Invariant Schrodinger and Dirac Wave Equations

The fact that the energy spectrum of equilibrium isotropic turbulence is given by *Planck* distribution (Figs. 6-10) is a strong evidence for quantum mechanical foundation of turbulence, [33, 34]. This is further supported by derivation of invariant *Schrödinger* equation from invariant *Bernoulli* equation in [34]. Hydrodynamic foundation of *Schrödinger* equation suggests that *Bohr* stationary states in quantum mechanics are connected to *stationary* sizes of clusters, *de Broglie* wave packets, in equilibrium fields. When both vorticity and iso-spin defined in Eq. (13) are zero one has a *Hamiltonian* non-dissipative system and for non-reactive incompressible fluid the flow field will be potential with $\mathbf{v}_\beta = -\nabla \Phi_\beta$ and the continuity Eq. (5) and *Cauchy* equation of motion (7) lead to invariant *Bernoulli* equation [34]

$$\frac{-\partial(\rho_\beta \Phi_\beta)}{\partial t} + \frac{[-\nabla(\rho_\beta \Phi_\beta)]^2}{2\rho_\beta} + p_\beta = \text{constant} = 0 \tag{202}$$

The constant in Eq. (202) is set to zero since pressure acts as a potential that is only defined to within an additive constant. Comparison of Eq. (202) with *Hamilton-Jacobi* equation of classical mechanics [2] written in invariant form

$$\frac{\partial S_\beta}{\partial t_\beta} + \frac{(\nabla S_\beta)^2}{2m_\beta} + U_\beta = 0 \tag{203}$$

resulted in the introduction of the invariant action [34, 153]

$$S_\beta(\mathbf{x}, t) = -\rho_\beta \Phi_\beta \tag{204}$$

The gradient of the action in Eq. (203) gives volumetric momentum density in harmony with the classical results [3]

$$\nabla S_\beta(\mathbf{x}, t) = -\rho_\beta \nabla \Phi_\beta = \rho_\beta \mathbf{v}_\beta = \boldsymbol{p}_\beta \tag{205}$$

In a recent study [34] it was shown that one can directly derive from the invariant *Bernoulli* equation (202) the invariant time-independent *Schrödinger* equation [34, 154, 155]

$$\nabla^2 \psi_\beta + \frac{8\pi^2 m_\beta}{h^2}(\bar{E}_\beta - \bar{U}_\beta)\psi_\beta = 0 \tag{206}$$

as well as the invariant time-dependent *Schrödinger* equation

$$i\hbar_{o\beta} \frac{\partial \Psi_\beta}{\partial t} + \frac{\hbar^2_\beta}{2m_\beta} \nabla^2 \Psi_\beta - \bar{U}_\beta \Psi_\beta = 0 \tag{207}$$

that governs the dynamics of particles from cosmic to tachyonic [34, 155] scales (Fig. 1). Since $\bar{E}_\beta = \bar{T}_\beta + \bar{U}_\beta$ (Ref. 34) *Schrödinger* equation (206) gives the *stationary states* of particles that are trapped within *de Broglie* wave packet under the *potential* acting as *Poincaré* stress. In view of the fact that pressure $U_\beta \equiv p_\beta = n_\beta \bar{U}_\beta$ [34] plays the role of potential in Eq. (206), anticipation of an *external pressure* or *stress* as being the cause of particle stability by *Poincaré* [117-119] is a testimony to the true genius of this great mathematician, physicist, and philosopher.

One may now introduce a new paradigm of the physical foundation of quantum mechanics according to which *Bohr* [72] stationary states will correspond to the statistically stationary sizes of atoms, *de Broglie* atomic wave packets, which will be governed by *Maxwell-Boltzmann* distribution function in Eq. (111) as shown in Fig. 13. Different energy levels of quantum mechanics are identified as different size atoms (elements). For example, in ESD field one views the transfer of a sub-particle (electron) from a small rapidly oscillating atom j to a large slowly oscillating atom i as transition from the high energy level j to the low energy level i, see Fig. 16, as schematically shown in Fig. 19.

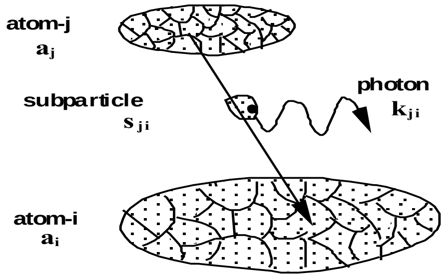


Fig. 19 Transition of electron e_{ij} from atom-j to atom-i leading to emission of photon k_{ij} [51].

Such a transition will be accompanied with emission of a "photon" that will carry away the excess energy [32, 34]

$$\Delta\varepsilon_{ji\beta} = \varepsilon_{j\beta} - \varepsilon_{i\beta} = h(\nu_{j\beta} - \nu_{i\beta}) \tag{208}$$

in harmony with *Bohr* [72] theory of *atomic* spectra. Therefore, the reason for the *quantum nature* of "atomic" energy spectra in equilibrium isotropic electrodynamics field is that transitions can only occur between atoms with energy levels that must satisfy the criterion of *stationarity* imposed by *Maxwell-Boltzmann* distribution function, [34, 139].

The results in equations (80) and (111) as well as Figs. 12, 16, and 19 suggest a generalized scale invariant description of transitions between energy levels of a statistical field at arbitrary scale β schematically shown in Fig. 20.

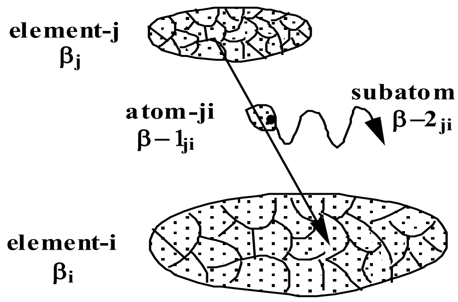


Fig. 20 Transition of "atom" a_{ij} from element-j to element-i leading to emission of sub-atomic particle s_{ij}.

According to Fig. 20, transition from high energy level j to low energy level i of "atomic" particle will lead to emission of a "sub-particle" at β−2 scale. If such emissions are induced or stimulated rather than spontaneous then one obtains coherent tachyon rays, neutrino rays, photon rays (laser), sub-particles rays (electron rays), atomic rays, molecular rays, . . . as discussed in Ref. 51.

For non-stationary relativistic fields, it was recently shown that by relating *Schrödinger* and *Dirac* wave functions as $\Psi_\beta = \tilde{\Psi}_\beta(z_j)\Psi_{D\beta}(x_j, t)$ with

$$\tilde{\Psi}_\beta = \exp[(\sqrt{i m_{\beta\text{-}1} w_{\beta\text{-}1}^2/\hbar})z_j] \tag{209}$$

, $\Psi_{D\beta} = \exp(i\alpha_m E_\beta t/\hbar)\ \exp(-i\alpha_m\alpha_j w x_j/\hbar)$, and the total energy defined as $E_{\beta\text{-}1} = 2m_{\beta\text{-}1}w_{\beta\text{-}1}^2$, one can derive from the equation of motion (24) the scale invariant relativistic wave equation [34]

$$i\hbar\left(\frac{1}{w_{\beta\text{-}1}}\frac{\partial\Psi_{D\beta}}{\partial t} + \alpha_j\frac{\partial\Psi_{D\beta}}{\partial x_j}\right) + (\alpha_m m_{\beta\text{-}1} w_{\beta\text{-}1})\Psi_{D\beta} = 0 \tag{210}$$

At the scale below *Casimir* [73] vacuum (ETD in Fig. 1) when $w_{\beta\text{-}1} = v_t = u_k = c$ is the speed of light, equation (210) becomes *Dirac* relativistic wave equation for electron [156, 34]

$$i\hbar\left(\frac{1}{c}\frac{\partial\Psi_{D\beta}}{\partial t} + \alpha_j\frac{\partial\Psi_{D\beta}}{\partial x_j}\right) + (\alpha_m mc)\Psi_{D\beta} = 0 \tag{211}$$

Therefore, the theory further described in [34] also provides a hydrodynamic foundation of *Dirac* relativistic wave equation for massive particles in the presence of spin.

XII. Concluding Remarks

A scale invariant model of statistical mechanics and its implications to the physical foundations of thermodynamics and kinetic theory of ideal gas were examined. *Boltzmann's* combinatoric method was employed to derive invariant forms of *Planck* energy and *Maxwell-Boltzmann* speed distribution functions. The impact of *Poincaré* recurrence theory on the problem of irreversibility in thermodynamics was discussed. The coincidence of normalized spacing between zeros of *Riemann* zeta function and normalized *Maxwell-Boltzmann* distribution and its connection to *Riemann* hypothesis were examined. Finally, hydrodynamic foundations of the invariant forms of both *Schrödinger* as well as *Dirac* wave equations were described. The universal nature of turbulence across broad range of spatio-temporal scales is in harmony with occurrence of fractals in physical science emphasized by *Takayasu* [157].

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